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J Nonlinear Sci
DOI 10.1007/s00332-010-9061-2

Journal of Nonlinear Science

Revisiting Energy Release Rates in Brittle Fracture

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Received: 29 September 2008 / Accepted: 19 February 2010

Abstract We revisit in a 2d setting the notion of energy release rate, which plays a pivotal role in brittle fracture. Through a blow-up method, we extend that notion to crack patterns which are merely closed sets connected to the crack tip. As an application, we demonstrate that, modulo a simple meta-stability principle, a moving crack cannot generically kink while growing continuously in time. This last result potentially renders obsolete in our opinion a longstanding debate in fracture mechanics on the correct criterion for kinking.

Keywords Brittle fracture · Energy release rate · Variational methods · Blow-up technique · Crack kink · Stability criterion

Mathematics Subject Classification (2000) 35R35 · 35J20 · 49S05 · 74R10 · 74A45 · 74G70

Communicated by A. Mielke.

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Published online: 29 April 2010

1 Introduction

Brittle fracture is by now "old" news in mechanics, and its foundation is considered by many as a closed subject. The basic mechanical principles governing quasi-static evolution, i.e., an evolution for which the effect of inertia is neglected, were postulated by Griffith (1920) about 90 years ago. Yet, they remain amazingly free of the usual stigmata of old age.

In essence, Griffith's formulation consists—in a 2-dimensional setting—in preassuming a crack path Γ and in computing for each crack length (the crack is assumed connected) the release of elastic energy associated with the infinitesimal extension of that crack. More precisely, if, say Ω is an elastic body and $\Gamma \subset \overline{\Omega}$, and if $u^0(t)$ is a boundary displacement applied on $\partial\Omega$, let $W_{\mathrm{el}}(t,l)$ denote the elastic energy associated with the elastic equilibrium of the body, with a crack of length l, submitted to the boundary displacement $u^0(t)$. Then the energy release rate associated with the crack length l at time t is given by

$$G(t,l) := -\lim_{h \searrow 0} \frac{1}{h}\big\{W_{\mathrm{el}}(t,l+h) - W_{\mathrm{el}}(t,l)\big\},$$

provided that limit exists. Of course, Griffith is not so preoccupied in Griffith (1920) with establishing conditions on both the crack path and the evolution under which one is at liberty to make such an assumption. Even today, haziness is the rule, and, to our knowledge, there are no precise results in the literature that explicit sufficient conditions on both the crack path and the evolution, and all the related results that we could locate are based on formal asymptotics.

In any case, Griffith then proceeds to motivate the existence of a positive constant k—often called fracture toughness, and to be viewed as the amount of energy released with each bond break for the underlying atomic lattice—such that

- $G(t,l(t)) \le k$;
- $l(t) \nearrow t$; and
- $\frac{\mathrm{d}l}{\mathrm{d}t}(t) \neq 0 \Rightarrow G(t,l(t)) = k$.

In other words, the energy release rate $G(t,l(t))$ is capped and the crack cannot move, unless the upper bound on that rate is met.

This three-pronged postulate provides the backbone of the theory of brittle fracture. A few years later, Fracture (1958) used a result previously established in Sneddon (1946) for a penny-shaped crack in an infinite domain with uniform stresses at infinity to establish that, for an isotropic material undergoing small deformations, the stress singularity at a crack tip is always in $1/\sqrt{r}$, where r is the distance to the tip, which led him to observe that, for a crack which is straight near its tip and points in the direction $\vec{e}$, the planar displacement field is always of the form $\sqrt{r}\{K_1\phi_1 + K_2\phi_2\}$, where ϕ_1, ϕ_2 are universal functions of the polar angle, while K_1, K_2, the stress intensity factors, contain information about the geometry and the loads; in our setting, we will sum up the dependency of the stress intensity factors upon the loads by the superscript t. Note that, if the stress field σ near the crack tip is pure traction, i.e., if $\sigma\vec{e}^{\perp} \parallel \vec{e}^{\perp}$ in a neighborhood of the tip, then $K_2 = 0$.

He then proceeded to compute the energy release rate along an extending straight crack originating at the boundary of Ω, and found that, for a crack of length l and, say displacement loads $u^0(t)$ on $\partial\Omega$,

$$G(t,l) = C\big\{\big(K_1^t\big)^2 + \big(K_2^t\big)^2\big\},$$

where C is explicitly given in terms of the elasticity of the material. Of course, here again, Irwin was not so interested in precise mathematical statements. On the one hand, establishing—and not a priori postulating—the exact nature of the singularity at the crack tip is not an easy task; we will refer to, e.g. Dauge (1988), Theorem 15.4 (referred to henceforth as simply Dauge 1988), for the appropriate result in our setting. On the other hand relating that singularity to a possible energy release rate requires tools of differentiation with respect to domain variation (Murat and Simon 1974). We refer the reader to Destuynder and Djaoua (1981) for the only precise setting we are aware of, that is, the case of a straight crack.

Remark 1.1 Many papers deal investigate the computations of the energy release rate. To our knowledge, the first rigorous proof of the existence of G, i.e., of the differentiable character of the potential energy as a function of crack length, is to be found in Destuynder and Djaoua (1981). As stated above, that proof has its root in the domain differentiation method. The underlying idea consists in mapping the evolving domain onto a fixed domain through a smooth enough transformation. As such, that method does not require a precise knowledge of the singularities at the crack tip. Consequently, it can be used independently of any symmetry restrictions on the elastic behavior; it extends to 3d domains; surface as well as body forces can be incorporated (with the corresponding change in the formula for G). It can also be extended to a nonlinear setting (see Knees and Mielke 2008). In Negri and Ortner (2008), the results of Destuynder and Djaoua (1981) are improved in that G is shown to be a continuous function of the crack length in the antiplane shear case.

However, those results cannot be used when a crack wishes to kink because of the lack of regularity of the transformation that would map the resulting domain onto a fixed domain.

Furthermore, linking G to the stress intensity factors, i.e., obtaining an Irwin type formula, necessitates a precise knowledge of the crack tip singularity. That knowledge seems to be lacking except for a linearly elastic material in 2d and when the crack is smooth and straight near its tip (see Dauge 1988 and also the generalized results in Costabel et al. 2003; see also Remark 2.1 below).

None of the above mentioned references address the ingredient that will be at the root of the present analysis: a general crack path allowing for nonsmooth extensions of the crack.

The mechanician involved in fracture mechanics is thus left to ponder the theoretical gear exhibited above, lamenting a remarkable yet incomplete toolkit. Actually, a mere counting of the number of unknowns versus equations makes it clear that the theory, as it stands, cannot predict crack path. So, for the last 50 years, mechanicians have attempted to import additional ingredients that would allow for such predictions.

The simplest setting is that of a straight crack that wishes to kink at a given time, that is to modify brutally its extensional direction. Assuming that the crack was propagating along the x-axis, we denote by ζ the kinking angle. Two competing criteria have been put forth. The first states that ζ will be such that the energy release rate at the time of extension of the crack from the kinking point is maximal among all possible straight add-cracks; this is referred to as the $G_{\max}$-criterion. The second postulates that ζ will be such that, after kinking, the limit, as the add-crack length tends to 0, of the stress intensity factor K_2 is 0; this is called the symmetry principle. Confusion is bound to arise because not only are those criteria essentially ad hoc, but also because they were shown in Amestoy and Leblond (1992) not to coincide.

The present study should be viewed as a contribution to the debate $G_{\max}$ versus $K_2 = 0$. We contend that, upon adoption of a general postulate of metastability of the total energy—the sum of the elastic and surface energies—with respect to connected add-cracks of small length, the debate is essentially pointless because there are *no* evolutions that kink along a “nice” geometric path—say with a $\mathcal{C}^1$ add-crack—while extending the crack continuously in time. This is the final result detailed in Proposition 4.6.

The suggested metastability postulate (see (4.13)) is simply stated in this paper and elaborated upon in Chambolle et al. (2009). It finds its root in the newly developed theory of variational fracture. We will not dwell upon that theory here and refer the interested reader to Bourdin et al. (2000) for a detailed exposition. However, please note that the first prong in Griffith’s criterion may be viewed as a first order necessary condition of metastability, because it states that the derivative of the total energy along “smooth” variations in the crack length must be nonnegative.

Our result is based on a precise computation, in a linear anisotropic setting, of the energy release rate associated with a large class of add-cracks (essentially, all connected add-cracks with finite length) and, in particular, add-cracks of density $\frac{1}{2}$ (in other words, of add-cracks that look like a line segment for small enough balls around the crack tip). This is the object of, first Lemma 2.6, then of Theorem 3.1, which combine to prove the existence of an energy release rate for such add-cracks in Corollary 3.7. As such, Theorem 3.1 may be viewed as the first rigorous computation of an energy release rate in a finite domain under arbitrary loads and for a large class of potential add-cracks, the consequence being that rate may be computed through the sole knowledge of the associated stress intensity factors through Irwin’s formula.

With that result at hand, we show that, in the more restrictive isotropic setting, line segment add-cracks cannot maximize the energy release rate in Theorem 4.1. That result appeals to difficult computations of expansions around the direction $\zeta = 0$ of the stress intensity factors associated with the kinking in a given direction ζ of a semiinfinite straight crack in $\mathbb{R}^2$. Those were performed in Amestoy and Leblond (1992). Unfortunately, the proof of Theorem 4.1 seems to require a knowledge of those coefficients for all values of ζ, or at least of a specific combination of those, see Conjecture 4.3. But this information cannot be derived from the sole results of Amestoy and Leblond (1992). We state the needed relation as a conjecture, observing that it is met near $\zeta = 0$ and that it is numerically evident. Since the conjecture is true if a specific nonzero universal entire function has no zeroes, we finally remark that, in the worst case scenario, there would be a finite number of universal kinking angles for which a time continuous evolution could take place, hardly what one should expect from a well-mannered kinking criterion.

We emphasize that, in all that follows, no attention is paid to the vexing issue of (linearized) non-interpenetration.

Finally note that we systematically omit, for the sake of notational simplicity, sets of 0-Lebesgue measure in writing integrals, i.e., if $\mathcal{L}_2(\Gamma)=0$, then $\int_{\Omega\setminus\Gamma} f\,\mathrm{d}x$ is written as $\int_\Omega f\,\mathrm{d}x$. Also, whenever $\varepsilon,\varepsilon'\in\mathcal{S}^{2\times 2}$, the space of symmetric 2×2-matrices, $\varepsilon\cdot\varepsilon'$ stands for $\mathrm{tr}(\varepsilon\varepsilon')$.

2 Linear Elasticity in a Cracked Domain—The Mathematical Setting

In all that follows:

A0. Ω is a Lipschitz bounded domain of $\mathbb{R}^2$ that contains the origin $O:=(0,0)$ and $\vec{e}_1,\vec{e}_2$ is a fixed orthonormal basis of $\mathbb{R}^2$.

The domain Ω is filled with a homogeneous elastic material with elasticity $\mathcal{C}$, a very strongly elliptic fourth order tensor with the usual symmetries of linear elasticity, i.e., a tensor such that $\mathcal{C}_{ijkh}=\mathcal{C}_{khij}=\mathcal{C}_{ikjh}$ and also such that $\mathcal{C}_{ijkh}\varepsilon_{ij}\varepsilon_{kh}>0$, $\forall\varepsilon_{ij}=\varepsilon_{ji}\neq 0$. We assume the existence of a precrack γ^i and will denote by Γ any additional crack, so that the compound crack will be $\gamma^i\cup\Gamma$.

As far as γ^i is concerned, the following is assumed:

A1. γ^i is a closed $\mathcal{C}^\infty$ curve in $\overline{\Omega}$ whose intersection with $\partial\Omega$ is at most a finite number of points;
A2. $\Omega\setminus\gamma^i$ is connected;
A3. the right endpoint of γ^i is the origin O;
A4. γ^i is a straight line segment in direction $\vec{e}_1$ in a neighborhood $B(O,\eta)$ of O.

In truth, it would be no essential restriction to assume, in lieu of Assumptions A1–A4, that γ^i is a straight line segment ending at O and originating either on the boundary $\partial\Omega$ (a notch), or inside Ω (a slit).

For a given displacement field $u^0\in H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^2)$, we wish to investigate the elastic equilibrium of $\Omega\setminus\gamma^i$ under the Dirichlet boundary condition u^0. To this end, we view u^0 as defined on all of $\mathbb{R}^2$: u^0 is then in $H^1(\mathbb{R}^2;\mathbb{R}^2)$ and, with no loss of generality, we may as well assume that it is compactly supported in $\mathbb{R}^2$. The solution (still denoted by u^0) of the elastic equilibrium of $\Omega\setminus\gamma^i$ under the Dirichlet boundary condition u^0 is the minimizer for

$$\min\left\{\frac{1}{2}\int_\Omega \mathcal{C}\epsilon(u)\cdot\epsilon(u)\,\mathrm{d}x : u\in H^1_{\mathrm{loc}}\big(\mathbb{R}^2\setminus\gamma^i;\mathbb{R}^2\big);\, u=u^0 \text{ in } \mathbb{R}^2\setminus\overline{\Omega}\right\}.$$

Note that the adoption of Dirichlet boundary conditions is unessential in what follows (see Remark 4.2).

We still denote by u^0 the unique solution; see, e.g. Chambolle (2003), Lemma 3. Note that it satisfies in particular

$$\begin{cases} -\operatorname{div}(\mathcal{C}\epsilon(u^0)) = 0, & \text{in } \Omega \setminus \gamma^i, \\ \mathcal{C}\epsilon(u^0)\nu = 0, & \text{on } \gamma^i \cap \Omega, \end{cases} \tag{2.1}$$

with ν any normal to $\partial(\Omega \setminus \gamma^i)$ at any point of the relative interior of $\gamma^i \cap \Omega$.

We will assume henceforth that, in the specific case at hand,

$$u^0 = \sqrt{|x|}\{K_1\phi_1 + K_2\phi_2\} + z := u^0_{\mathrm{O}} + z, \tag{2.2}$$

with $z \in H^2(B(O,\eta/2) \setminus \gamma^i; \mathbb{R}^2) \cap H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \gamma^i; \mathbb{R}^2)$; ϕ_1, ϕ_2 are universal functions that only depend on the polar angle at O and on $\mathcal{C}$, and K_1, K_2 depend on $\Omega, \mathcal{C}, u^0, \gamma^i$. The constants K_1, K_2 are called the *stress intensity factors*, and it is not our purpose here to describe them in any details, referring the interested reader to, e.g. Amestoy and Leblond (1992). The following remark demonstrates that assumption (2.2) certainly holds true at least in the isotropic case.

Remark 2.1 The conditions under which decomposition (2.2) is valid are not so easily found in the literature in spite of a widespread belief in the square root nature of the singularity at the crack tip. The investigation of the singular behavior of the elastic field at corners is certainly a popular field; we refer to, e.g. Dauge (1988) for a treatise on that topic, although there are many other relevant available references. However, it is quite a challenge to locate a precise result like that invoked above.

For the sake of completeness, we briefly indicate in the present remark how to derive (2.2) in the isotropic case, i.e., when

$$\mathcal{C} = \lambda i \otimes i + 2\mu I,$$

with i the identity matrix on $\mathbb{R}^2$, I that on $\mathcal{S}^{2\times 2}$, and λ, μ the classical Lamé coefficients of isotropic elasticity.

To do this, we will appeal to a clearly stated result in Grisvard (1989) which addresses the case of a bounded polygonal domain with a straight crack in a 2d isotropic setting (see Théorème I, Remarque 1.2, Théorème 6.1, and Remarque 6.4 in Grisvard 1989). Starting with a weak solution u^0 of (2.1) in that case, we introduce a radial cut-off $\zeta(r)$ with $\zeta \equiv 0$ for $r > \eta/2$. Then it is an easy consequence of the fact that $u^0 \in H^1(B(O,\eta); \mathbb{R}^2)$ that $w_\zeta := \zeta u^0$ satisfies

$$\begin{cases} -\operatorname{div}(\mathcal{C}\epsilon(w_\zeta)) = f_\zeta, & \text{in } \Omega \setminus \gamma^i, \\ \mathcal{C}\epsilon(w_\zeta)\nu = g_\zeta, & \text{on } \gamma^i \cap \Omega, \\ w_\zeta \equiv 0, & \text{in } B(O,\eta) \setminus \overline{B(O,\eta/2)}, \end{cases}$$

with $f_\zeta \in L^2(B(O,\eta); \mathbb{R}^2)$, $g_\zeta \in H^{\frac{1}{2}}(\gamma^i; \mathbb{R}^2)$, with $\operatorname{supp} g_\zeta \subset \gamma^i \cap (B(O,3\eta/4) \setminus \overline{B(O,\eta/4)})$. Thus, w_ζ is also solution to

$$\begin{cases} -\operatorname{div}(\mathcal{C}\epsilon(w_\zeta)) = f_\zeta, & \text{in } G, \\ \mathcal{C}\epsilon(w_\zeta)\nu = g_\zeta, & \text{on } \partial G, \end{cases}$$

where G is a polygon made up of the union of a segment $[O, M] \subset \gamma^i$ with $M \in B(O, \eta) \setminus \overline{B(O, 3\eta/4)}$—we emphasize that $[O, M]$ is indeed a segment in view of Assumption A1—and of a convex polygon with boundary inside $B(O, \eta) \setminus \overline{B(O, 3\eta/4)}$. Since $f_\zeta \in L^2(G; \mathbb{R}^2)$ while $g_\zeta \in H^{\frac{1}{2}}(\partial G; \mathbb{R}^2)$, while condition (1.5) of Théorème I in Grisvard (1989) (precisely written in Remark 7.2.2.5 in Grisvard 1985) are trivially satisfied since g_ζ is null near all corner points. Application of that theorem yields the desired result.

That (2.2) also holds true in the anisotropic case is undoubtedly true, although we were unable to pinpoint the relevant result in our admittedly cursory perusal of the literature. Many papers resort to formal asymptotics for a justification of (2.1); see, for example Argatov and Nazarov (2002), where the authors *a priori* assume a power series expansion of the elastic solution at the crack tip in a 2d anisotropic setting.

Remark 2.2 Define

$$\Gamma^i := \mathbb{R}^- \vec{e}_1,$$

set

$$u^0_\varepsilon(y) := \frac{u^0(\varepsilon y) - u^0(0)}{\sqrt{\varepsilon}}, \qquad z_\varepsilon(y) := \frac{z(\varepsilon y) - z(0)}{\sqrt{\varepsilon}}$$

and note that

$$u^0_\varepsilon(y) = u^0_{\mathrm{O}}(y) + z_\varepsilon(y).$$

Then for all $r > 0$,

$$\begin{aligned} u^0_\varepsilon \to u^0_{\mathrm{O}}, \quad & \text{uniformly in } B(O, r) \setminus \Gamma^i \text{ and} \\ & \text{strongly in } W^{1,p}\big(\mathbb{R}^2 \setminus \Gamma^i; \mathbb{R}^2\big),\ 1 \le p < \infty. \end{aligned} \tag{2.3}$$

Indeed, by classical Sobolev injections, $\nabla z \in L^s(B(O, \eta/2) \setminus \gamma^i; \mathbb{R}^2)$, $\forall s < \infty$, so that, recalling Assumption A4 on γ^i, for ε so small that $\varepsilon r < \eta/2$,

$$\begin{aligned} \int_{B(O,r)\setminus\Gamma^i} |\nabla z_\varepsilon|^p \,\mathrm{d}y &= \varepsilon^{p/2} \int_{B(O,r)\setminus\Gamma^i} \big|\nabla z(\varepsilon y)\big|^p \,\mathrm{d}y \\ &= \varepsilon^{p/2-2} \int_{B(O,\varepsilon r)\setminus\gamma^i} \big|\nabla z(y)\big|^p \,\mathrm{d}y \\ &\le \varepsilon^{p/2-2} \big(\pi \varepsilon^2 r^2\big)^{(q-1)/q} \left(\int_{B(O,\eta/2)\setminus\gamma^i} \big|\nabla z(y)\big|^{pq} \,\mathrm{d}y \right)^{1/q} \\ &= C_r \varepsilon^{p/2-2/q}, \end{aligned}$$

with C_r depending only on r. So, for any $p < \infty$, we can choose q large enough so that

$$\nabla z_\varepsilon \to 0, \quad \text{strongly in } L^p\big(B(O, r) \setminus \Gamma^i; \mathbb{R}^2\big). \tag{2.4}$$

Since z_ε is a fortiori in $L^p(B(O,r)\setminus\Gamma^i;\mathbb{R}^2)$ for ε small enough, Morrey's inequality implies, upon choosing p large enough in (2.4), that for all r's,

$$z_\varepsilon - z_\varepsilon(0) \to 0, \quad \text{uniformly on } B(O,r)\setminus\Gamma^i.$$

But $z_\varepsilon(0)=0$, hence (2.3).

Remark 2.3 Note that u^0_{O} satisfies

$$\begin{cases} -\operatorname{div}(\mathcal{C}\epsilon(u^0_{\mathrm{O}}))=0, & \text{in } \mathbb{R}^2\setminus\Gamma^i,\\ \mathcal{C}\epsilon(u^0_{\mathrm{O}})\nu=0, & \text{on } \Gamma^i. \end{cases}$$

We now wish to add a crack Γ "at the crack tip". We assume that

A5. Γ is a compact connected set in Ω with $\mathcal{H}^1(\Gamma)<\infty$;
A6. $O\in\Gamma$.

We henceforth define, for any point $M\in\mathbb{R}^2$, $\mathcal{A}^{\mathrm{M}}$ as *the set of* Γ*'s that satisfy Assumptions* A5, A6 with M in lieu of O.

Then as before, we wish to investigate the elastic equilibrium of $\Omega\setminus(\gamma^i\cup\Gamma)$ under the Dirichlet boundary condition u^0. We denote by u^Γ a solution of

$$\min\left\{\frac{1}{2}\int_\Omega \mathcal{C}\epsilon(u)\cdot\epsilon(u)\,\mathrm{d}x : u\in H^1_{\mathrm{loc}}\big(\Omega\setminus\big(\gamma^i\cup\Gamma\big);\mathbb{R}^2\big), u=u^0 \text{ on } \partial\Omega\right\}. \tag{2.5}$$

Under Assumptions A5 and A6, proving existence of such a displacement is not an issue—see Chambolle (2003)—while uniqueness is true if and only if $\Omega\setminus(\gamma^i\cup\Gamma)$ is connected. Otherwise, u^Γ is any rigid motion inside each connected component of $\Omega\setminus(\gamma^i\cup\Gamma)$ which does not touch $\partial\Omega$.

We define, for all $\Gamma\in\mathcal{A}^{\mathrm{O}}$,

$$\mathcal{F}^{\gamma^i}(\Gamma) := \frac{1}{2}\int_\Omega\big(\mathcal{C}\epsilon\big(u^\Gamma\big)\cdot\epsilon\big(u^\Gamma\big)-\mathcal{C}\epsilon\big(u^0\big)\cdot\epsilon\big(u^0\big)\big)\,\mathrm{d}x. \tag{2.6}$$

The following estimate holds true

Lemma 2.4 *There exists two nonnegative constants* $\overline{G}$ *and* $\underline{G}$ *such that, for* $l>0$ *small,*

$$-\overline{G}+o(1)\le\frac{1}{l}\inf_\Gamma\big\{\mathcal{F}^{\gamma^i}(\Gamma):\Gamma\in\mathcal{A}^{\mathrm{O}};\mathcal{H}^1(\Gamma)\le l\big\}\le-\underline{G}+o(1). \tag{2.7}$$

Proof First consider $\Gamma=[O,I]\cup\partial B(O,l/(2\pi+1))$, with $\overrightarrow{OI}:=-l/(2\pi+1)\vec{e}_1$ (so that $\Gamma\cup\gamma^i$ looks like γ^i, together with a circle of radius $l/(2\pi+1)$ centered at O). Then in view of (2.2),

$$\mathcal{F}^{\gamma^i}(\Gamma)\le-\frac{1}{2}\int_{B(O,l/(2\pi+1))}\mathcal{C}\epsilon\big(u^0\big)\cdot\epsilon\big(u^0\big)\,\mathrm{d}x\sim-\underline{G}l+o(l),$$

with

$$\underline{G} := \frac{1}{2(2\pi+1)} \int_{B(O,1)} \mathcal{C}\epsilon\big(u_{\mathrm{O}}^0\big) \cdot \epsilon\big(u_{\mathrm{O}}^0\big)\,\mathrm{d}x.$$

Now, with an argument identical to that developed in Chambolle et al. (2008), Sect. 4 (see, in particular, (4.3) of that reference), we obtain the following inequality for any $\Gamma \in \mathcal{A}^{\mathrm{O}}$

$$\mathcal{F}^{\gamma^i}(\Gamma) \ge -C \int_{\Omega} \big|\tau - \mathcal{C}\epsilon\big(u^0\big)\big|^2\,\mathrm{d}x, \tag{2.8}$$

for all $\tau \in L^2(\Omega \setminus \gamma^i; \mathbb{R}^2 \times \mathbb{R}^2)$ symmetric and such that

$$\begin{aligned} &\int_{\Omega} \tau \cdot \epsilon(w)\,\mathrm{d}x = 0, \quad \forall w \in H^1_{\mathrm{loc}}\big(\mathbb{R}^2 \setminus \big(\gamma^i \cup \Gamma\big); \mathbb{R}^2\big) \text{ with} \\ &\qquad w \equiv 0 \text{ on } \mathbb{R}^2 \setminus \overline{\Omega} \text{ and } \epsilon(w) \in L^2\big(\mathbb{R}^2 \setminus \big(\gamma^i \cup \Gamma\big); \mathcal{S}^{2\times 2}\big). \end{aligned} \tag{2.9}$$

The only difference with (4.3) in Chambolle et al. (2008) is that Ω in that reference has to be replaced with $\Omega \setminus \gamma^i$ here, which is no restriction in view of Assumptions A1 and A2.

Note that $\Gamma \subset B(O,l)$ since $\mathcal{H}^1(\Gamma) \le l$ and $O \in \Gamma$. Consider $\varphi \in \mathcal{C}^\infty(\mathbb{R}^2)$ with $\varphi \equiv 1$ outside $B(O, 7l/4)$ and $\varphi \equiv 0$ inside $B(O, 5l/4)$. Take $\tau \equiv \mathcal{C}\epsilon(u^0)$ outside $B(O,2l)$, $\tau \equiv 0$ in $B(O,l)$ and $\tau \equiv \epsilon(v) + \varphi\mathcal{C}\epsilon(u^0)$ in $B(O,2l) \setminus \overline{B}(O,l)$ with

$$\begin{cases} \operatorname{div}\epsilon(v) = -\operatorname{div}(\varphi\mathcal{C}\epsilon(u^0)) & \text{in } B(O,2l) \setminus (\overline{B}(O,l) \cup \gamma^i) \\ \epsilon(v) \cdot \nu = 0 & \text{on } \partial B(O,l) \cup \partial B(O,2l) \cup \gamma^i, \end{cases}$$

with ν the exterior normal to $B(O,2l) \setminus \overline{B}(O,l)$, or the normal to γ^i. Then τ satisfies (2.9).

In view of the assumed regularity of γ^i, an elementary integration by parts establishes that

$$\int_{B(O,2l)\setminus\overline{B}(O,l)} \big|\epsilon(v)\big|^2\,\mathrm{d}x \le C \int_{B(O,2l)} \big|\epsilon\big(u^0\big)\big|^2\,\mathrm{d}x,$$

so that, in the end, for that particular choice of τ,

$$\int_{\Omega} \big|\tau - \mathcal{C}\epsilon\big(u^0\big)\big|^2\,\mathrm{d}x \le C \int_{B(O,2l)} \big|\epsilon\big(u^0\big)\big|^2\,\mathrm{d}x.$$

Recalling (2.8) and appealing once again to (2.2), we finally obtain

$$\mathcal{F}^{\gamma^i}(\Gamma) \ge -\overline{G}l + o(l),$$

with

$$\overline{G} := C \int_{B(O,2)} \big|\epsilon\big(u_{\mathrm{O}}^0\big)\big|^2\,\mathrm{d}x. \qquad \square$$

Remark 2.5 Note that $\overline{G}, \underline{G}$ vanish if $u_{\mathrm{O}}^{0} = 0$ (and are positive otherwise). *We assume from now on that* $u_{\mathrm{O}}^{0} \neq 0$, or still that

$$K_1 \text{ or } K_2 \neq 0, \tag{2.10}$$

that is, the load actually induces a singularity at the crack tip.

Finally, the following lemma holds true.

Lemma 2.6 *Assume that* $\Gamma(l) \in \mathcal{A}^{\mathrm{O}}$ *also satisfies*

$$\Gamma(l) \subset \Gamma(l'), \quad l \leq l'; \qquad \lim_{l \searrow 0} \frac{\mathcal{H}^1(\Gamma(l))}{l} = 1, \tag{2.11}$$

and that, for some l_0, $\Gamma(l_0)$ *satisfies*

A7. Γ has density $\frac{1}{2}$ at O, i.e.,

$$\lim_{s \to 0} \frac{\mathcal{H}^1(\Gamma \cap B(O, s))}{2s} = \frac{1}{2}.$$

Then there exists a sequence $\{l_j \searrow 0\}$ *and a line segment* $[O, M]$ *with* $|\overrightarrow{OM}| = 1$ *such that*

$$\Gamma_j := \frac{1}{l_j} \Gamma(l_j) \text{ converge in the sense of Hausdorff to } [O, M].$$

Further, all possible such "blow-up limits" are unit length line segments.

Proof In view of Assumptions A5, A6, and of the second item in property (2.11), Blaschke's selection theorem (i.e., the compactness of equi-bounded compact connected sets for the Hausdorff distance; see Falconer 1985, Theorem 3.16) proves the existence of a sequence $\{l_j \searrow 0\}$, and of a compact connected set Γ_0 such that

$$\Gamma_j \text{ converge in the sense of Hausdorff to } \Gamma_0, \quad \text{with } O \in \Gamma_0.$$

Further, by application of Golab's theorem (i.e., the lower semicontinuity of the 1-dimensional Haudorff measure for compact connected sets converging for the Hausdorff distance; see e.g. Falconer 1985) we also have

$$\mathcal{H}^1(\Gamma_0) \leq 1. \tag{2.12}$$

In view of the above, Γ_0 will be of the announced form $[O, M]$, provided that we show that $\Gamma_0 \cap \partial B(O, 1) \neq \emptyset$.

To that effect, we consider $t < 1$ and remark that, if, for a subsequence of $\{l_j\}$ still indexed by j, $\Gamma(l_j) \subset B(O, tl_j)$, then, in view of Assumption A7 and of the ordering property in (2.11), for any $l_0 > l > 0$,

$$
\begin{aligned}
1 &= \lim_j \frac{\mathcal{H}^1(\Gamma(l) \cap B(O, tl_j))}{tl_j} \\
&\geq \limsup_j \frac{\mathcal{H}^1(\Gamma(l_j) \cap B(O, tl_j))}{tl_j} \\
&= \frac{1}{t} \limsup_j \mathcal{H}^1\big((\Gamma_j)\big) = \frac{1}{t},
\end{aligned} \tag{2.13}
$$

clearly a contradiction. Thus, for all $t < 1$, $\Gamma(l_j) \setminus B(O, tl_j) \neq \emptyset$ for j large enough. Consider $x_j \in \Gamma(l_j) \setminus B(O, tl_j)$; $x_j/l_j \notin B(O, t)$. But, since $O \in \Gamma_j$ and Γ_j is connected with length less than $1 + o(1/j)$, $x_j/l_j \in \overline{B}(O, 1 + o(1/j))$. Thus, a subsequence of $\{x_j/l_j\}$ converges to some point $x \in \overline{B}(O, 1) \setminus B(O, t)$ which also belongs to Γ_0 because of the Hausdorff convergence of Γ_j to Γ_0. Thus, $\Gamma_0 \cap (\overline{B}(O, 1) \setminus B(O, t))$ is not empty and the result is achieved upon letting t tend to 1.

Since all possible blow-up limits satisfy (2.12), the last statement of the theorem also follows. □

Remark 2.7 Consider a connected add-crack Γ with Γ of density $\frac{1}{2}$ at O. Then upon setting $\Gamma(l) := \Gamma \cap B(O, l)$, Lemma 2.6 demonstrates that blow up limits of density $\frac{1}{2}$ connected add-cracks at the crack tip are line segments of length 1, a fact which is obvious if investigating add-cracks that are smooth, in which case $\Gamma(l)$ can be taken to be the connected subarc of Γ of arc length l with $O \in \Gamma(l)$.

Remark that, in the nonsmooth case, various blow-up subsequences may in general converge to different unit length line segments. Just take $\Gamma = \exp(-t^2)(\cos t\vec{e}_1 + \sin t\vec{e}_2)$, $t \in [0, \infty]$, which has no well-defined tangent as $t \nearrow \infty$ and for which M will be any element of $\partial B(O, 1)$.

3 Blow-up Limits of a Converging Sequence of Finite Length Sets

In this section, we prove a general blow-up result (Theorem 3.1) on a converging sequence of compact connected sets containing the origin O, this in the context of Assumptions A0–A6 of Sect. 2. We then apply this result in Corollary 3.7 to the specific sequence Γ_j constructed in Lemma 2.6.

Theorem 3.1 *Assume that Γ_ε is a Hausdorff-converging sequence of elements of $\mathcal{A}^{\mathrm{O}}$, with $\sup_\varepsilon \mathcal{H}^1(\Gamma_\varepsilon) < \infty$. Then recalling definition (2.6) of $\mathcal{F}^{\gamma^i}$,*

$$
\lim_\varepsilon \frac{1}{\varepsilon} \mathcal{F}^{\gamma^i}(\varepsilon\Gamma_\varepsilon) = \mathcal{F}^{\Gamma^i}(\Gamma), \tag{3.1}
$$

where Γ is the Hausdorff limit of Γ_ε and

$$\mathcal{F}^{\Gamma^i}(\Gamma) := \min\Bigg\{\frac{1}{2}\int_{\mathbb{R}^2} \mathcal{C}\epsilon(w)\cdot\epsilon(w)\,\mathrm{d}x + \int_{B(O,r)} \mathcal{C}\epsilon\big(u^0_{\mathrm{O}}\big)\cdot\epsilon(w)\,\mathrm{d}x - \int_{\partial B(O,r)} \mathcal{C}\epsilon\big(u^0_{\mathrm{O}}\big) : (w\otimes\nu)\,d\mathcal{H}^1 : w \in H^1_{\mathrm{loc}}\big(\mathbb{R}^2\setminus\big(\Gamma^i\cup\Gamma\big)\big), \epsilon(w)\in L^2\big(\mathbb{R}^2\big)\Bigg\}. \tag{3.2}$$

In (3.2), $r>0$ *is any radius such that* $\Gamma\subset B(O,r)$.

In other words, $\frac{1}{\varepsilon}\mathcal{F}^{\gamma^i}(\varepsilon\cdot)$ *converges continuously to* $\mathcal{F}^{\Gamma^i}$.

Remark 3.2 Note that it is easily seen that the definition of $\mathcal{F}^{\Gamma^i}$ above is independent of r. Actually, if $\hat{w}$ is a solution to the associated Euler equation—see (3.17) below—, then

$$\mathcal{F}^{\Gamma^i}(\Gamma) = -\frac{1}{2}\int_{\mathbb{R}^2} \mathcal{C}\epsilon(\hat{w})\cdot\epsilon(\hat{w})\,\mathrm{d}x. \tag{3.3}$$

Note also that the thesis of Theorem 3.1 still holds if the load u_0 is applied to only part of the boundary $\partial_D\Omega$ of $\partial\Omega$, or in the case of a “soft device,” that is, if the boundary conditions on some part of $\partial\Omega$ are of the form $C\epsilon(u)\nu = g$, g being a surface force density.

Remark 3.3 The above convergence is trivially stronger than Γ-convergence. Thus, if Γ_ε are minimizers—or almost minimizers, up to an error that goes to 0 with ε—of $\frac{1}{\varepsilon}\mathcal{F}^{\gamma^i}(\varepsilon\cdot)$, under the constraint $\mathcal{H}^1(\Gamma_\varepsilon)\le 1$, then the limits Γ (in the Hausdorff sense) of converging subsequences Γ_{ε_j} are minimizers of $\mathcal{F}^{\Gamma^i}$ under the same constraint. In particular, since (2.7) also reads as

$$-\overline{G} \le \frac{1}{\varepsilon}\inf_{\Gamma}\big\{\mathcal{F}^{\gamma^i}(\varepsilon\Gamma) : \Gamma\in\mathcal{A}^{\mathrm{O}}; \mathcal{H}^1(\Gamma)\le 1\big\} \le -\underline{G}, \tag{3.4}$$

and since the set

$$\mathcal{A}^{\mathrm{O}}_1 := \big\{\Gamma\in\mathcal{A}^{\mathrm{O}} : \mathcal{H}^1(\Gamma)\le 1\big\}, \tag{3.5}$$

is sequentially compact for the topology associated to Hausdorff convergence in view of Blaschke’s selection criterion, together with Golab’s theorem, we deduce that the limit of the infimum in (2.7) as $l\searrow 0$ exists and is equal to

$$-\mathcal{G}_1 := \min_{\Gamma\in\mathcal{A}^{\mathrm{O}}_1} \mathcal{F}^{\Gamma^i}.$$

Remark 3.4 $\mathcal{F}^{\Gamma^i}(\Gamma)$ defines a generalized energy release rate, associated to a particular crack pattern Γ. Note, however, that if the minimizing Γ in Remark 3.3 is not homogeneous, i.e., if $\lambda\Gamma\notin\Gamma$ for all $\lambda<1$, it may not be so that a continuously growing path can follow that pattern, and the interpretation of the associated release rate is more delicate; see Sect. 4 for a more in depth investigation of maximal energy release rates.

Proof of Theorem 3.1 First, fix $r > 0$ such that $\Gamma \subset B(O, r)$, and observe that, for ε small enough, $\Gamma_\varepsilon \subset B(O, r)$. By definition,

$$\frac{1}{\varepsilon}\mathcal{F}^{\gamma^i}(\varepsilon\Gamma_\varepsilon) = \frac{1}{\varepsilon}\min\bigg\{\frac{1}{2}\int_\Omega \big(\mathcal{C}\epsilon(u)\cdot\epsilon(u) - \mathcal{C}\epsilon\big(u^0\big)\cdot\epsilon\big(u^0\big)\big)\,\mathrm{d}x : \\ u \in H^1_{\mathrm{loc}}\big(\Omega \setminus \big(\gamma^i \cup \varepsilon\Gamma_\varepsilon\big); \mathbb{R}^2\big), u = u^0 \text{ on } \partial\Omega\bigg\}, \tag{3.6}$$

and, by definition of $u^{\varepsilon\Gamma_\varepsilon}$ (see (2.5)),

$$\frac{1}{\varepsilon}\mathcal{F}^{\gamma^i}(\varepsilon\Gamma_\varepsilon) = \frac{1}{2\varepsilon}\int_\Omega \big(\mathcal{C}\epsilon\big(u^{\varepsilon\Gamma_\varepsilon}\big)\cdot\epsilon\big(u^{\varepsilon\Gamma_\varepsilon}\big) - \mathcal{C}\epsilon\big(u^0\big)\cdot\epsilon\big(u^0\big)\big)\,\mathrm{d}x.$$

The change of variable $w_\varepsilon := u^{\varepsilon\Gamma_\varepsilon} - u^0$ transforms the above expression into

$$-\frac{1}{2\varepsilon}\int_\Omega \mathcal{C}\epsilon(w_\varepsilon)\cdot\epsilon(w_\varepsilon)\,\mathrm{d}x + \frac{1}{\varepsilon}\int_\Omega \mathcal{C}\epsilon\big(u^{\varepsilon\Gamma_\varepsilon}\big)\cdot\epsilon(w_\varepsilon)\,\mathrm{d}x.$$

Now, it is straightforward from (2.5) that $u^{\varepsilon\Gamma_\varepsilon}$ satisfies the equation

$$\int_\Omega \mathcal{C}\epsilon\big(u^{\varepsilon\Gamma_\varepsilon}\big)\cdot\epsilon(v)\,\mathrm{d}x = 0 \tag{3.7}$$

for any $v \in H^1_{\mathrm{loc}}(\Omega \setminus (\gamma^i \cup \varepsilon\Gamma_\varepsilon); \mathbb{R}^2)$ with $\epsilon(v) \in L^2(\Omega; \mathcal{S}^{2\times 2})$ and $v = 0$ on $\partial\Omega$. Further, w_ε is an admissible test for (3.7), so that

$$\int_\Omega \mathcal{C}\epsilon\big(u^{\varepsilon\Gamma_\varepsilon}\big)\cdot\epsilon(w_\varepsilon)\,\mathrm{d}x = 0,$$

and thus

$$\frac{1}{\varepsilon}\mathcal{F}^{\gamma^i}(\varepsilon\Gamma_\varepsilon) = -\frac{1}{2\varepsilon}\int_\Omega \mathcal{C}\epsilon(w_\varepsilon)\cdot\epsilon(w_\varepsilon)\,\mathrm{d}x. \tag{3.8}$$

Recalling the lower bound in (3.4), we conclude, in view of (3.8), that

$$\frac{1}{2}\int_\Omega \mathcal{C}\epsilon(w_\varepsilon)\cdot\epsilon(w_\varepsilon)\,\mathrm{d}x \le \overline{G}\varepsilon. \tag{3.9}$$

Now, the change of variable $w := u - u^0$ permits one to rewrite the integral in (3.6) as

$$\frac{1}{2}\int_\Omega \mathcal{C}\epsilon(w)\cdot\epsilon(w) + \int_\Omega \mathcal{C}\epsilon\big(u^0\big)\cdot\epsilon(w)\,\mathrm{d}x.$$

But $w = 0$ on $\partial\Omega$ and, for ε small enough, $\varepsilon\Gamma_\varepsilon \subset B(O, r\varepsilon)$, so that, since u^0 satisfies (2.1),

$$\int_\Omega \mathcal{C}\epsilon\big(u^0\big)\cdot\epsilon(w)\,\mathrm{d}x = \int_{B(O,r\varepsilon)} \mathcal{C}\epsilon\big(u^0\big)\cdot\epsilon(w)\,\mathrm{d}x - \int_{\partial B(O,r\varepsilon)} \mathcal{C}\epsilon\big(u^0\big)\cdot(w\otimes\nu)\,\mathrm{d}\mathcal{H}^1,$$

where ν is the exterior normal to the disc $B(O, r\varepsilon)$. Consequently, we get that

$$\frac{1}{\varepsilon}\mathcal{F}^{\gamma^i}(\varepsilon\Gamma_\varepsilon) = \frac{1}{\varepsilon}\min\left\{\frac{1}{2}\int_\Omega \mathcal{C}\epsilon(w)\cdot\epsilon(w)\,\mathrm{d}x + \int_{B(O,r\varepsilon)} \mathcal{C}\epsilon\big(u^0\big)\cdot\epsilon(w)\,\mathrm{d}x \right.$$
$$- \int_{\partial B(O,r\varepsilon)} \mathcal{C}\epsilon\big(u^0\big)\cdot(w\otimes\nu)\,\mathrm{d}\mathcal{H}^1 :$$
$$\left. w \in H^1_{\mathrm{loc}}\big(\Omega\setminus\big(\gamma^i\cup\varepsilon\Gamma_\varepsilon\big);\mathbb{R}^2\big),\, w=0 \text{ on } \partial\Omega\right\}, \tag{3.10}$$

with, by construction, w_ε as one of the minimizers.

Set, for $y \in \Omega/\varepsilon$, $\hat{w}_\varepsilon(y) = w_\varepsilon(\varepsilon y)/\sqrt{\varepsilon}$. By an appropriate rescaling of the integrals (replacing w by $w(\varepsilon y)/\sqrt{\varepsilon}$), we find that, in the notation of Remark 2.2,

$$\frac{1}{\varepsilon}\mathcal{F}^{\gamma^i}(\varepsilon\Gamma_\varepsilon) = \min\left\{\frac{1}{2}\int_{\Omega/\varepsilon} \mathcal{C}\epsilon(w)\cdot\epsilon(w)\,\mathrm{d}x + \int_{B(O,r)} \mathcal{C}\epsilon\big(u^0_\varepsilon\big)\cdot\epsilon(w)\,\mathrm{d}x \right.$$
$$- \int_{\partial B(O,r)} \mathcal{C}\epsilon\big(u^0_\varepsilon\big)\cdot(w\otimes\nu)\,\mathrm{d}\mathcal{H}^1 :$$
$$\left. w \in H^1_{\mathrm{loc}}\left(\left(\frac{\Omega\setminus\gamma^i}{\varepsilon}\right)\Big\backslash\Gamma_\varepsilon;\mathbb{R}^2\right),\, w=0 \text{ on } \partial\Omega/\varepsilon\right\}, \tag{3.11}$$

and that $\hat{w}_\varepsilon$ is a minimizer. We also deduce from (3.9) that

$$\int_{\Omega/\varepsilon} \mathcal{C}\epsilon(\hat{w}_\varepsilon)\cdot\epsilon(\hat{w}_\varepsilon)\,\mathrm{d}x \le C. \tag{3.12}$$

Hence, up to possible subsequence extraction, we may assume that $\epsilon(\hat{w}_\varepsilon)$—extended by 0 outside Ω/ε—converges weakly in $L^2(\mathbb{R}^2;\mathcal{S}^{2\times 2})$ to some field of symmetric matrices. Because of the Hausdorff convergence of Γ_ε to Γ and since γ^i/ε is a line segment on any ball centered at O for ε small enough, Poincaré–Korn's inequality applied to any compactly contained open smooth subset of $\mathbb{R}^2\setminus(\Gamma^i\cup\Gamma)$ demonstrates that there exists $\hat{w}\in H^1_{\mathrm{loc}}(\mathbb{R}^2\setminus(\Gamma^i\cup\Gamma);\mathbb{R}^2)$ such that

$$\epsilon(\hat{w}_\varepsilon) \rightharpoonup \epsilon(\hat{w}), \quad \text{weakly in } L^2\big(\mathbb{R}^2;\mathcal{S}^{2\times 2}\big),$$

while

$$\hat{w}_\varepsilon + r_\varepsilon \to \hat{w}, \quad \text{strongly in } L^2_{\mathrm{loc}}\big(\mathbb{R}^2\setminus\big(\Gamma^i\cup\Gamma\big);\mathbb{R}^2\big),$$

where r_ε is an ε-dependent rigid body displacement in each connected component of $((\Omega\setminus\gamma^i)/\varepsilon)\setminus\Gamma$.

Note for future reference that, by L^2-weak lower semi-continuity of $\epsilon\mapsto\int_{\mathbb{R}^2}\mathcal{C}\epsilon\cdot\epsilon\,\mathrm{d}x$, we deduce from (3.12) that

$$\int_{\mathbb{R}^2} \mathcal{C}\epsilon(\hat{w})\cdot\epsilon(\hat{w})\,\mathrm{d}x \le C. \tag{3.13}$$

But it is immediately checked that the energy in (3.11) is invariant if any rigid displacement is added to w in each connected component. Indeed, the first two terms are trivially unchanged, while the third term is also unchanged upon integration by parts on $((\Omega \setminus \gamma^i)/\varepsilon) \setminus B(O, r)$ and in view of (2.1) appropriately rescaled.

Thus, we may assume, without loss of generality, that

$$\begin{cases} \hat{w}_\varepsilon \to \hat{w}, & \text{strongly in } L^2_{\text{loc}}(\mathbb{R}^2 \setminus (\Gamma^i \cup \Gamma); \mathbb{R}^2) \\ \epsilon(\hat{w}_\varepsilon) \rightharpoonup \epsilon(\hat{w}), & \text{weakly in } L^2(\mathbb{R}^2; \mathcal{S}^{2\times 2}) \\ \quad \hat{w}_\varepsilon \text{ is a rigid body displacement outside } \Omega/\varepsilon. \end{cases} \tag{3.14}$$

Since $\hat{w}_\varepsilon$ is a minimizer in (3.11), it satisfies the following weak Euler–Lagrange equation

$$\int_{\Omega/\varepsilon} \mathcal{C}\epsilon(\hat{w}_\varepsilon) \cdot \epsilon(v)\,\mathrm{d}x = -\int_{B(O,r)} \mathcal{C}\epsilon\big(u^0_\varepsilon\big) \cdot \epsilon(v)\,\mathrm{d}x + \int_{\partial B(O,r)} \mathcal{C}\epsilon\big(u^0_\varepsilon\big) : (v \otimes \nu)\,\mathrm{d}\mathcal{H}^1 \tag{3.15}$$

for every

$$v \in H^1_{\text{loc}}\left(\left(\frac{\Omega \setminus \gamma^i}{\varepsilon}\right) \Big\backslash \Gamma_\varepsilon; \mathbb{R}^2\right)$$

with $\epsilon(v) \in L^2(\mathbb{R}^2; \mathcal{S}^{2\times 2})$ and v a rigid displacement on $\partial\Omega/\varepsilon$. Consequently,

$$\frac{1}{\varepsilon}\mathcal{F}^{\gamma^i}(\varepsilon\Gamma_\varepsilon) = \frac{1}{2}\left\{\int_{B(O,r)} \mathcal{C}\epsilon\big(u^0_\varepsilon\big) \cdot \epsilon(\hat{w}_\varepsilon)\,\mathrm{d}x - \int_{\partial B(O,r)} \mathcal{C}\epsilon\big(u^0_\varepsilon\big) \cdot (\hat{w}_\varepsilon \otimes \nu)\,\mathrm{d}\mathcal{H}^1\right\}.$$

In view of convergences (2.3), (3.14), we obtain that $\mathcal{C}\epsilon(u^0_\varepsilon)\nu$ converges strongly in $H^{-\frac{1}{2}}(\partial B(O, r); \mathbb{R}^2)$ and $\hat{w}_\varepsilon$ weakly in $H^{\frac{1}{2}}(\partial B(O, r); \mathbb{R}^2)$ and so we can pass to the limit in the expression above. We obtain

$$\lim_{\varepsilon} \frac{1}{\varepsilon}\mathcal{F}^{\gamma^i}(\varepsilon\Gamma_\varepsilon) = \frac{1}{2}\left\{\int_{B(O,r)} \mathcal{C}\epsilon\big(u^0_{\mathrm{O}}\big) \cdot \epsilon(\hat{w})\,\mathrm{d}x - \int_{\partial B(O,r)} \mathcal{C}\epsilon\big(u^0_{\mathrm{O}}\big) \cdot (\hat{w} \otimes \nu)\,\mathrm{d}\mathcal{H}^1\right\}, \tag{3.16}$$

which is shown in the same way to be equal to $\mathcal{F}^{\Gamma^i}(\Gamma)$, *provided we show first that* $\hat{w}$ is a minimizer for the problem in (3.2). This is true if and only if we show that $\hat{w}$ satisfies the Euler equation for the minimization of the integral in the definition of $\mathcal{F}^{\Gamma^i}(\Gamma)$, i.e., iff

$$\int_{\mathbb{R}^2} \mathcal{C}\epsilon(\hat{w}) \cdot \epsilon(v)\,\mathrm{d}x = -\int_{B(O,r)} \mathcal{C}\epsilon\big(u^0_{\mathrm{O}}\big) \cdot \epsilon(v)\,\mathrm{d}x + \int_{\partial B(O,r)} \mathcal{C}\epsilon\big(u^0_{\mathrm{O}}\big) : (v \otimes \nu)\,\mathrm{d}\mathcal{H}^1 \tag{3.17}$$

for any $v \in H^1_{\text{loc}}(\mathbb{R}^2 \setminus (\Gamma^i \cup \Gamma); \mathbb{R}^2)$ with $\epsilon(v) \in L^2(\mathbb{R}^2; \mathcal{S}^{2\times 2})$.

We may as well assume that the support of the test function v in (3.17) is bounded. Indeed, if $\varphi(x)$ is a smooth function with support in $B(O, 2)$ and equal to 1 on $B(O, 1)$, we set, for $R > 0$ large enough,

$$v_R(x) = \varphi\left(\frac{x}{R}\right)\big(v(x) - A_R x - b_R\big)$$

where $A_R x + b_R$ is a rigid displacement such that the following Poincaré–Korn inequality holds true:

$$\|v - A_R x - b_R\|_{L^2(B(O,2R)\setminus B(O,R);\mathbb{R}^2)} \le CR\|\epsilon(v)\|_{L^2(B(O,2R)\setminus B(O,R);\mathcal{S}^{2\times 2})}. \quad (3.18)$$

Then a.e. in $\mathbb{R}^2$,

$$\epsilon(v_R)(x) = \varphi\left(\frac{x}{R}\right)\epsilon(v) + \frac{1}{R}\nabla\varphi\left(\frac{x}{R}\right) \odot \big(v(x) - A_R x - b_R\big),$$

so, thanks to (3.18), $\epsilon(v_R)$ converges strongly in $L^2(\mathbb{R}^2; \mathcal{S}^{2\times 2})$ to $\epsilon(v)$ as $R \to \infty$.

Then we may also assume that $v \in H^1(\mathbb{R}^2 \setminus (\Gamma^i \cup \Gamma); \mathbb{R}^2)$, with bounded support. Indeed, Theorem 1 in Chambolle (2003) states in particular that, given any test function $v \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus (\Gamma^i \cup \Gamma); \mathbb{R}^2)$, with $\epsilon(v) \in L^2(\mathbb{R}^2; \mathcal{S}^{2\times 2})$ and support inside $B(O, R)$ for some (large) R, there exists a sequence $\{v_n\}$ of displacements in $H^1(B(O, R + 1) \setminus (\Gamma^i \cup \Gamma); \mathbb{R}^2)$ with $\epsilon(v_n) \to \epsilon(v)$ in $L^2(B(O, R + 1); \mathcal{S}^{2\times 2})$. Observe that, possibly subtracting rigid displacements, v_n converges strongly to 0 in $L^2(B(O, R + 1) \setminus \overline{B}(O, R); \mathbb{R}^2)$. If φ is a smooth cut-off function equal to 1 on $B(O, R)$ and with support in $B(O, R + 1)$, $v'_n = v_n\varphi$ has bounded support and is such that $\epsilon(v'_n) \to \epsilon(v)$ strongly in $L^2(B(O, R + 1); \mathcal{S}^{2\times 2})$.

We have therefore shown that it is enough to consider in (3.17) test displacements v which are in $H^1(\mathbb{R}^2 \setminus (\Gamma^i \cup \Gamma); \mathbb{R}^2)$—in lieu of $H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus (\Gamma^i \cup \Gamma); \mathbb{R}^2)$—and vanish outside some large ball $B(O, R)$, $R > r > 0$.

Since Γ_ε converges in the sense of Hausdorff to Γ, $[(\Omega \setminus \gamma^i)/\varepsilon \setminus \Gamma_\varepsilon \cap B(O, R+1)]$ converges in the sense of the complementary Hausdorff topology to $\mathbb{R}^2 \setminus (\Gamma^i \cup \Gamma) \cap B(O, R + 1)$, so that, according to, e.g., Lemma 3.4 in Bucur and Varchon (2000), there exist functions $v_\varepsilon \in H^1((\Omega \setminus \gamma^i)/\varepsilon \setminus \Gamma_\varepsilon; \mathbb{R}^2)$ such that

$$\begin{cases} v_\varepsilon \to v, & \text{strongly in } L^2(B(O, R + 1); \mathbb{R}^2) \\ \nabla v_\varepsilon \to \nabla v, & \text{strongly in } L^2(B(O, R + 1); \mathbb{R}^{2\times 2}), \end{cases}$$

where the gradients are extended by 0 outside their natural domain of definition. Since $v_\varepsilon \to 0$ strongly in $L^2(B(O, R + 1) \setminus B(O, R); \mathbb{R}^2)$, the same truncation by φ as in the previous paragraph implies that $v'_\varepsilon = \varphi v_\varepsilon$ are functions in $H^1((\Omega \setminus \gamma^i)/\varepsilon_j \setminus \Gamma_j; \mathbb{R}^2)$ which vanish on $\mathbb{R}^2 \setminus B(O, R + 1)$, and are such that $v_\varepsilon \to v$, $\nabla v_\varepsilon \to \nabla v$, strongly in $L^2(\mathbb{R}^2; \mathbb{R}^2)$. Moreover, for ε small enough, $B(O, R + 1) \subset \Omega/\varepsilon$ so that each v_ε is an admissible test functions for (3.15).

We pass to the limit with $v = v_\varepsilon$ in (3.15) and deduce that (3.17) holds. Hence the right-hand side of (3.16) is $\mathcal{F}^{\Gamma^i}(\Gamma)$ and Theorem 3.1 is proved. In particular, although we had to consider a subsequence to assert the convergence of $\hat{w}_\varepsilon$ to some limit, the corresponding limit of $\frac{1}{\varepsilon}\mathcal{F}^{\gamma^i}(\varepsilon\Gamma_\varepsilon)$ is independent of the choice of this subsequence and, therefore, the convergence (3.1) of the whole sequence is established. □

Theorem 3.1 also applies to the case where $\Omega = \mathbb{R}^2$, with γ^i replaced by $\Gamma^i \cup [O, M]$ where $\overrightarrow{OM} = \vec{e}$ ($\vec{e}$, a unit vector in $\mathbb{R}^2$), provided that we adopt (3.10) as definition for $\frac{1}{\varepsilon}\mathcal{F}^{\gamma^i}(\varepsilon\Gamma)$, with the appropriate change in the test fields. Specifically, we obtain the following.

Theorem 3.5 *Assume that Γ_ε is a Hausdorff-converging sequence of elements of $\mathcal{A}^{\mathrm{M}}$, with $\sup_\varepsilon \mathcal{H}^1(\Gamma_\varepsilon) < \infty$, and denote by Γ its Hausdorff limit. Then defining*

$$\mathcal{F}_\infty^{\Gamma^i\cup[O,M]}(\varepsilon\Gamma_\varepsilon) := \left(\mathcal{F}^{\Gamma^i}\left([O,M]\cup\varepsilon\Gamma_\varepsilon\right) - \mathcal{F}^{\Gamma^i}\left([O,M]\right)\right),$$

we have

$$\lim_\varepsilon \frac{1}{\varepsilon}\mathcal{F}_\infty^{\Gamma^i\cup[O,M]}(\varepsilon\Gamma_\varepsilon) = \mathcal{F}^{\Gamma_{[O,M]}}(\Gamma).$$

The definition of $\mathcal{F}^{\Gamma_{[O,M]}}(\Gamma)$ is identical to that of $\mathcal{F}^{\Gamma^i}(\Gamma)$, provided that O is replaced by M as the origin, Γ^i is replaced by $\Gamma_{[O,M]} := \mathbb{R}^-\vec{e}$, and the displacement field u^0_{O} by u^0_{M} defined as follows. Consider

$$\hat{u}^0 := \hat{w} + u^0_{\mathrm{O}}, \tag{3.19}$$

where $\hat{w}$ is the solution to (3.17) with $\Gamma = [O,M]$. Then as in (2.2), $\hat{u}^0 = u^0_{\mathrm{M}} + z$, where

$$u^0_{\mathrm{M}} = \sqrt{|x|}\left(K_1^M\phi_1^M + K_2^M\phi_2^M\right);$$

here $|x|$ is the distance from the point M, ϕ_1^M and ϕ_2^M depend only on the polar angle with respect to the direction $\vec{e}$, and z is an H^2-function in a neighborhood of the point M.

Proof For $\Gamma_\varepsilon \in \mathcal{A}^{\mathrm{M}}$ converging in the sense of Hausdorff to Γ, we consider, for R large enough so that $[O,M]\cup\varepsilon\Gamma_\varepsilon \subset B(O,R)$, with $\mathcal{F}^{\Gamma^i}$ defined in (3.2),

$$\begin{aligned}
\mathcal{F}_\infty^{\Gamma^i\cup[O,M]}(\varepsilon\Gamma_\varepsilon) &= \mathcal{F}^{\Gamma^i}\left([O,M]\cup\varepsilon\Gamma_\varepsilon\right) - \mathcal{F}^{\Gamma^i}\left([O,M]\right)\\
&= \min\Bigg\{\frac{1}{2}\int_{\mathbb{R}^2}\mathcal{C}\epsilon(w)\cdot\epsilon(w)\,\mathrm{d}x + \int_{B(O,R)}\mathcal{C}\epsilon\left(u^0_{\mathrm{O}}\right)\cdot\epsilon(w)\,\mathrm{d}x\\
&\qquad - \int_{\partial B(O,R)}\mathcal{C}\epsilon\left(u^0_{\mathrm{O}}\right)\cdot(w\otimes\nu)\,\mathrm{d}\mathcal{H}^1 - \frac{1}{2}\int_{\mathbb{R}^2}\mathcal{C}\epsilon(\hat{w})\cdot\epsilon(\hat{w})\,\mathrm{d}x\\
&\qquad - \int_{B(O,R)}\mathcal{C}\epsilon\left(u^0_{\mathrm{O}}\right)\cdot\epsilon(\hat{w})\,\mathrm{d}x\\
&\qquad + \int_{\partial B(O,R)}\mathcal{C}\epsilon\left(u^0_{\mathrm{O}}\right)\cdot(\hat{w}\otimes\nu)\,\mathrm{d}\mathcal{H}^1 :\\
&\qquad w \in H^1_{\mathrm{loc}}\left(\mathbb{R}^2\setminus\left(\Gamma^i\cup[O,M]\cup\varepsilon\Gamma_\varepsilon\right);\mathbb{R}^2\right)\Bigg\}.
\end{aligned} \tag{3.20}$$

Then setting in turn $\overline{w} := w - \hat{w}$, simple algebra transforms (3.20) into

$$
\begin{aligned}
\mathcal{F}_\infty^{\Gamma^i \cup [O,M]}(\varepsilon \Gamma_\varepsilon) = \min\Bigg\{ & \frac{1}{2}\int_{\mathbb{R}^2} \mathcal{C}\epsilon(\overline{w}) \cdot \epsilon(\overline{w})\,\mathrm{d}x + \int_{B(O,R)} \mathcal{C}\epsilon\big(\hat{u}^0\big) \cdot \epsilon(\overline{w})\,\mathrm{d}x \\
& - \int_{\partial B(O,R)} \mathcal{C}\epsilon\big(\hat{u}^0\big) \cdot (\overline{w} \otimes \nu)\,\mathrm{d}\mathcal{H}^1 \\
& + \frac{1}{2}\int_{\mathbb{R}^2 \setminus B(O,R)} \mathcal{C}\epsilon(\hat{w}) \cdot \epsilon(\overline{w})\,\mathrm{d}x \\
& + \int_{\partial B(O,R)} \mathcal{C}\epsilon(\hat{w}) \cdot (\overline{w} \otimes \nu)\,\mathrm{d}\mathcal{H}^1 : \\
& \overline{w} \in H^1_{\mathrm{loc}}\big(\mathbb{R}^2 \setminus \big(\Gamma^i \cup [O,M] \cup \varepsilon\Gamma_\varepsilon\big); \mathbb{R}^2\big)\Bigg\}.
\end{aligned} \tag{3.21}
$$

Now, for any $\overline{w}$ as in (3.21),

$$
\frac{1}{2}\int_{\mathbb{R}^2 \setminus B(O,R)} \mathcal{C}\epsilon(\hat{w}) \cdot \epsilon(\overline{w})\,\mathrm{d}x + \int_{\partial B(O,R)} \mathcal{C}\epsilon(\hat{w}) \cdot (\overline{w} \otimes \nu)\,\mathrm{d}\mathcal{H}^1 = 0.
$$

Indeed, according to an argument identical to that used at the end the proof of Theorem 3.1, it suffices to check this equality for $\overline{w} \in H^1(\mathbb{R}^2 \setminus (\Gamma^i \cup [O,M] \cup \varepsilon\Gamma_\varepsilon); \mathbb{R}^2)$, with compact support. But the equality holds true for such $\overline{w}$'s because, according to (3.17),

$$
-\operatorname{div}\big(\mathcal{C}\epsilon(\hat{w})\big) = 0, \quad \text{in } \mathbb{R}^2 \setminus B(O,R),\ R \text{ large enough.}
$$

Thus,

$$
\begin{aligned}
\mathcal{F}_\infty^{\Gamma^i \cup [O,M]}(\varepsilon \Gamma_\varepsilon) = \min\Bigg\{ & \frac{1}{2}\int_{\mathbb{R}^2} \mathcal{C}\epsilon(\overline{w}) \cdot \epsilon(\overline{w})\,\mathrm{d}x + \int_{B(O,R)} \mathcal{C}\epsilon\big(\hat{u}^0\big) \cdot \epsilon(\overline{w})\,\mathrm{d}x \\
& - \int_{\partial B(O,R)} \mathcal{C}\epsilon\big(\hat{u}^0\big) \cdot (\overline{w} \otimes \nu)\,\mathrm{d}\mathcal{H}^1 : \\
& \overline{w} \in H^1_{\mathrm{loc}}\big(\mathbb{R}^2 \setminus \big(\Gamma^i \cup [O,M] \cup \varepsilon\Gamma_\varepsilon\big); \mathbb{R}^2\big)\Bigg\}.
\end{aligned} \tag{3.22}
$$

Now, if r is such that $\Gamma_\varepsilon \subset B(M,r)$ for ε small enough, which is possible since Γ_ε converges to Γ in the sense of Hausdorff, then, if ε is also small enough so that $B(M, r\varepsilon) \subset B(O,R)$,

$$
\begin{aligned}
&\int_{B(O,R) \setminus B(M,r\varepsilon)} \mathcal{C}\epsilon\big(\hat{u}^0\big) \cdot \epsilon(\overline{w})\,\mathrm{d}x \\
&\quad = \int_{\partial B(O,R)} \mathcal{C}\epsilon\big(\hat{u}^0\big) \cdot (\overline{w} \otimes \nu)\,\mathrm{d}\mathcal{H}^1 - \int_{\partial B(M,r\varepsilon)} \mathcal{C}\epsilon\big(\hat{u}^0\big) \cdot (\overline{w} \otimes \nu)\,\mathrm{d}\mathcal{H}^1.
\end{aligned} \tag{3.23}
$$

Indeed, thanks to Remark 2.3 and to (3.17), and because in the case of a line segment, $\hat{u}^0 \in H^1(B(O,R) \setminus (\Gamma^i \cup [O,M])$,

$$-\operatorname{div}\big(\mathcal{C}\epsilon(\hat{u}^0)\big) = 0, \quad \text{ in } B(O,R) \setminus B(M, r\varepsilon).$$

Collecting (3.22), (3.23), we obtain

$$\frac{1}{\varepsilon}\Big(\mathcal{F}^{\Gamma^i}\big([O,M] \cup \varepsilon\Gamma_\varepsilon\big) - \mathcal{F}^{\Gamma^i}\big([O,M]\big)\Big) = \frac{1}{\varepsilon}\mathcal{F}_\infty^{\Gamma^i \cup [O,M]}(\varepsilon\Gamma_\varepsilon),$$

where, for $\Gamma \in \mathcal{A}^{\mathrm{M}}$, $\mathcal{F}_\infty^{\Gamma^i \cup [O,M]}(\Gamma)$ can be defined, for any r such that $\Gamma \subset B(M,r)$, as

$$\begin{aligned}\mathcal{F}_\infty^{\Gamma^i \cup [O,M]}(\Gamma) := \min\Bigg\{ & \frac{1}{2}\int_{\mathbb{R}^2} \mathcal{C}\epsilon(\overline{w}) \cdot \epsilon(\overline{w})\,\mathrm{d}x + \int_{B(M,r)} \mathcal{C}\epsilon\big(\hat{u}^0\big) \cdot \epsilon(\overline{w})\,\mathrm{d}x \\ & - \int_{\partial B(M,r)} \mathcal{C}\epsilon\big(\hat{u}^0\big) \cdot (\overline{w} \otimes \nu)\,\mathrm{d}\mathcal{H}^1 : \\ & \overline{w} \in H^1_{\mathrm{loc}}\big(\mathbb{R}^2 \setminus \big(\Gamma^i \cup [O,M] \cup \Gamma\big); \mathbb{R}^2\big)\Bigg\}. \end{aligned} \tag{3.24}$$

In particular, when $\Gamma = \varepsilon\Gamma_\varepsilon$, we can replace r by εr.

Remark that $\frac{1}{\varepsilon}\mathcal{F}_\infty^{\Gamma^i \cup [O,M]}(\varepsilon\Gamma_\varepsilon)$ also reads as

$$\frac{1}{\varepsilon}\mathcal{F}_\infty^{\Gamma^i \cup [O,M]}(\varepsilon\Gamma_\varepsilon) = -\frac{1}{2\varepsilon}\int_{\mathbb{R}^2} \mathcal{C}\epsilon\big(w^{\Gamma^i \cup [O,M] \cup \varepsilon\Gamma_\varepsilon}\big) \cdot \epsilon\big(w^{\Gamma^i \cup [O,M] \cup \varepsilon\Gamma_\varepsilon}\big)\,\mathrm{d}x,$$

where $w^{\Gamma^i \cup [O,M] \cup \varepsilon\Gamma_\varepsilon}$ is a solution to the associated Euler equation, hence upon the usual rescaling

$$\begin{cases} \check{w}_\varepsilon(y) := w^{\Gamma^i \cup [O,M] \cup \varepsilon\Gamma_\varepsilon}(M + \varepsilon y)/\sqrt{\varepsilon}, \\ \hat{u}^0_\varepsilon(y) := \hat{u}^0(M + \varepsilon y)/\sqrt{\varepsilon}, \end{cases}$$

as

$$\frac{1}{\varepsilon}\mathcal{F}_\infty^{\Gamma^i \cup [O,M]}(\varepsilon\Gamma_\varepsilon) = -\frac{1}{2}\int_{\mathbb{R}^2} \mathcal{C}\epsilon(\check{w}_\varepsilon) \cdot \epsilon(\check{w}_\varepsilon)\,\mathrm{d}x,$$

where $\check{w}_\varepsilon$ is a solution to

$$\int_{\mathbb{R}^2} \mathcal{C}\epsilon(\check{w}_\varepsilon) \cdot \epsilon(v)\,\mathrm{d}x = -\int_{B(M,r)} \mathcal{C}\epsilon\big(\hat{u}^0_\varepsilon\big) \cdot \epsilon(v)\,\mathrm{d}x + \int_{\partial B(M,r)} \mathcal{C}\epsilon\big(\hat{u}^0_\varepsilon\big) : (v \otimes \nu)\,\mathrm{d}\mathcal{H}^1,$$

with test functions $v \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus ((N^\varepsilon + \Gamma^i) \cup [N^\varepsilon, M] \cup \Gamma_\varepsilon); \mathbb{R}^2)$ with $\epsilon(v) \in L^2(\mathbb{R}^2; \mathcal{S}^{2\times 2})$ and where N^ε is defined through $\overrightarrow{MN^\varepsilon} = \frac{1}{\varepsilon}\overrightarrow{MO}$.

The proof of Theorem 3.1 for $\mathcal{F}_\infty^{\Gamma^i \cup [O,M]}$ is identical, provided that the lower bound estimate in (3.4) still holds true in this new setting. To this effect, we have to establish the analogue of (2.8) in the present context, i.e.,

$$\mathcal{F}_\infty^{\Gamma^i \cup [O,M]}(\Gamma) \geq -C\int_{\mathbb{R}^2} |\tau|^2\,\mathrm{d}x, \tag{3.25}$$

for all τ with $\tau \in L^2(\mathbb{R}^2 \setminus (\varGamma^i \cup [O,M]); \mathbb{R}^2 \times \mathbb{R}^2)$ symmetric such that

$$\int_{\mathbb{R}^2} \tau \cdot \epsilon(\overline{w})\,\mathrm{d}x + \int_{B(M,r)} \mathcal{C}\epsilon\big(\hat{u}^0\big) \cdot \epsilon(\overline{w})\,\mathrm{d}x - \int_{\partial B(M,r)} \mathcal{C}\epsilon\big(\hat{u}^0\big) \cdot (\overline{w} \otimes \nu)\,\mathrm{d}\mathcal{H}^1 = 0, \tag{3.26}$$

for all $\overline{w} \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus (\varGamma^i \cup [O,M] \cup \varGamma); \mathbb{R}^2)$ with $\epsilon(\overline{w}) \in L^2(\mathbb{R}^2 \setminus (\varGamma^i \cup [O,M] \cup \varGamma); \mathcal{S}^{2\times 2})$.

To establish (3.25), we simply note that, in view of (3.24), convex duality permits to rewrite the expression $\mathcal{F}_\infty^{\varGamma^i \cup [O,M]}(\varGamma)$ as

$$\begin{aligned}\mathcal{F}_\infty^{\varGamma^i \cup [O,M]}(\varGamma) = \max_\tau \Bigg[&-\frac{1}{2}\int_{\mathbb{R}^2} \mathcal{C}^{-1}\tau \cdot \tau\,\mathrm{d}x \\ &+ \min_{\overline{w}} \bigg\{ \int_{\mathbb{R}^2} \tau \cdot \epsilon(\overline{w})\,\mathrm{d}x + \int_{B(M,r)} \mathcal{C}\epsilon\big(\hat{u}^0\big) \cdot \epsilon(\overline{w})\,\mathrm{d}x \\ &- \int_{\partial B(M,r)} \mathcal{C}\epsilon\big(\hat{u}^0\big) \cdot (\overline{w} \otimes \nu)\,\mathrm{d}\mathcal{H}^1 \bigg\} \Bigg].\end{aligned}$$

Clearly, the minimum is $-\infty$ unless τ is such that

$$\int_{\mathbb{R}^2} \tau \cdot \epsilon(\overline{w})\,\mathrm{d}x + \int_{B(M,r)} \mathcal{C}\epsilon\big(\hat{u}^0\big) \cdot \epsilon(\overline{w})\,\mathrm{d}x - \int_{\partial B(M,r)} \mathcal{C}\epsilon\big(\hat{u}^0\big) \cdot (\overline{w} \otimes \nu)\,\mathrm{d}\mathcal{H}^1 = 0,$$

hence the sought result (3.25).

Now, consider $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ with $\varphi \equiv 1$ inside $B(M,r)$ and $\equiv 0$ outside $B(M,2r)$. Take $\tau \equiv \mathcal{C}\epsilon(\check{w})$ in $B(M,r)$, with $\check{w}$ a minimizer in the definition (3.24) of $\mathcal{F}_\infty^{\varGamma^i \cup [O,M]}(\varGamma)$, and $\tau \equiv \epsilon(v) + \varphi\mathcal{C}\epsilon(\check{w})$ in $B(O,2r) \setminus \overline{B}(O,r)$ with

$$\begin{cases} \operatorname{div}\epsilon(v) = -\operatorname{div}(\varphi\mathcal{C}\epsilon(\check{w})) & \text{in } B(M,2r) \setminus (\overline{B}(M,r) \cup \varGamma^i \cup [O,M]) \\ \epsilon(v)\cdot\nu = 0 & \text{on } \partial B(M,2r) \cup \partial B(M,r) \cup \varGamma^i \cup [O,M] \end{cases}$$

with ν the exterior normal to $B(M,2r) \setminus \overline{B}(M,r)$, or the normal to $\varGamma^i \cup [O,M]$. Then, τ satisfies (3.26) and is thus an admissible test in inequality (3.25).

From this point on, the proof of the lower bound is similar to that in Lemma 2.4. The remainder of the proof is identical to that of Theorem 3.1. □

In the setting of Lemma 2.6, consider a sequence $\{l_j \searrow 0\}$ such that $\frac{1}{l_j}\varGamma(l_j)$ converges in the sense of Hausdorff with $j \nearrow \infty$. Then we adopt the following generalization of the classical energy release rate to a path that belongs to $\mathcal{A}^{\mathrm{O}}$ (instead of being a smooth curve).

Definition 3.6 The energy release rate associated to $\varGamma(l_j)$ is the limit, if it exists of $-\frac{1}{l_j}\mathcal{F}^{\gamma^i}(\varGamma(l_j))$.

According to Theorem 3.1, that limit does exist, and, if all Hausdorff limits of Hausdorff converging subsequences are identical (in the smooth add-crack case of

Remark 2.7, for example), then there is only one energy release rate, namely,

$$\lim_{l \searrow 0} -\frac{1}{l} \mathcal{F}^{\gamma^i}\big(\Gamma(l)\big).$$

Combining Lemma 2.6 with Theorem 3.1, we immediately obtain the following.

Corollary 3.7 *Assume that $\Gamma(l) \in \mathcal{A}^{\mathrm{O}}$ also satisfies Assumption* A7 *for some l_0, as well as* (2.11). *Then the energy release rate associated with* (*a Hausdorff converging sequence of*) *$\Gamma(l)/l$ is given by $-\mathcal{F}^{\Gamma^i}([O, M])$ where $[O, M]$, with $|\overrightarrow{OM}| = 1$, is the corresponding Hausdorff limit* (*of that sequence*).

Remark 3.8 All segments $[O, M]$ with $M \in \partial B(O, 1)$ can be attained as Hausdorff limits of a sequence of $\Gamma(l)$ with $\Gamma(l) \in \mathcal{A}^{\mathrm{O}}$, satisfying (2.11) and Assumption A7 for each l, as is obviously demonstrated by taking $\Gamma(l) := l[O, M]$.

In the light of the previous corollary and of Remark 3.3, it is natural to investigate

1. The nature of the minimizers for

$$\min_{\Gamma \in \mathcal{A}^{\mathrm{O}}_1} \big\{\mathcal{F}^{\Gamma^i}(\Gamma)\big\},$$

   that we now know do exist. In particular, are unit length line segments among those?
2. The value of the maximal energy release rate among all possible $\Gamma(l) \in \mathcal{A}^{\mathrm{O}}$ that also satisfy (2.11), but not Assumption A7.

We address these in the next section, at least in the isotropic case; see Remark 2.1.

4 Maximal Energy-Release Rates

Our first result provides a complete answer, albeit generically in the negative, to the first question formulated at the close of the previous section. To this effect, we further specialize (2.10) in Remark 2.5 to the case where $K_2 \neq 0$. This is the most interesting case because it is "universally" believed that, when $K_2 = 0$,

$$\mathcal{G}_1 = \mathcal{G}_{\mathrm{clas}}\Big(:= \min_{M \in \partial B(O,1)} \big\{\mathcal{F}^{\Gamma^i}\big([O, M]\big)\big\}\Big) = -\mathcal{F}^{\Gamma^i}\big([O, M_0]\big), \quad \text{with}$$
$$[O, M_0] = \vec{e}_1. \tag{4.1}$$

The result is the object of the following theorem, whose proof is conditional upon a conjecture on the transfer matrix between stress intensity coefficients (see Conjecture 4.3 below). In Remark 4.4, we explain why we believe that the conjecture is correct and what could occur, should this conjecture fail.

Also, we have been unable however to locate a precise reference that proves (4.1). For our part, we prove this in Remark 4.5, conditionally upon yet another conjecture on the transfer matrix.

Theorem 4.1 *The notation is that of Theorem 3.1 and of Remark 3.3. Then provided that*

$$K_2 \neq 0 \quad in\ (2.2), \tag{4.2}$$

and also that Conjecture 4.3 holds true, then

$$-\mathcal{G}_1 = \min_{\Gamma \in \mathcal{A}_1^{\mathrm{O}}} \{\mathcal{F}^{\Gamma^i}(\Gamma)\} < \min_{M \in \partial B(O,1)} \{\mathcal{F}^{\Gamma^i}([O,M])\} := -\mathcal{G}_{\mathrm{clas}},$$

where $\mathcal{A}_1^{\mathrm{O}}$ was defined in (3.5).

Proof Consider a point $M \in \partial B(O,1)$ such that $\mathcal{F}^{\Gamma^i}([O,M]) := -\mathcal{G}_{\mathrm{clas}}$. Note that such a point exists because, if $M_n \in \partial B(O,1)$ is a infimizing sequence, then a subsequence, still indexed by n converges to $M \in \partial B(O,1)$. According to Remark 3.8, $[O,M_n]$ is attained as a Hausdorff limit, so that, in view of (3.12), $\hat{w}_n$, the solution of the Euler equation (3.17) associated with $\mathcal{F}^{\Gamma^i}([O,M_n])$ satisfies (3.13). But then, $\epsilon(\hat{w}_n)$ converges to some $\epsilon(\hat{w})$ weakly in $L^2(\mathbb{R}^2;\mathcal{S}^{2\times 2})$ with $\hat{w} \in L^2_{\mathrm{loc}}(\mathbb{R}^2 \setminus (\Gamma^i \cup [O,M]);\mathbb{R}^2)$. For every test function $v \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus (\Gamma^i \cup [O,M]);\mathbb{R}^2)$ with $\epsilon(v) \in L^2(\mathbb{R}^2;\mathcal{S}^{2\times 2})$, it is a simple matter to construct a sequence v_n of test functions in $H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus (\Gamma^i \cup [O,M_n]);\mathbb{R}^2)$ with $\epsilon(v) \in L^2(\mathbb{R}^2;\mathcal{S}^{2\times 2})$—simply write v, then v_n in the form $v(r,\theta-\theta_M)$, resp. $v(r,\theta-\theta_{M_n})$ with obvious notation—so that both v_n and ∇v_n converge strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$. Passing to the limit in (3.17) with $\Gamma = [O,M_n]$ and $v = v_n$, we conclude that $\hat{w}$ satisfies (3.17) with $\Gamma = [O,M]$.

Now consider $\frac{1}{\varepsilon}\mathcal{F}_\infty^{\Gamma^i \cup [O,M]}(\varepsilon[M,N])$ with $N \in \partial B(M,1)$. Then according to Theorem 3.5,

$$\frac{1}{\varepsilon}\mathcal{F}_\infty^{\Gamma^i \cup [O,M]}(\varepsilon[M,N]) \to \mathcal{F}^{\Gamma_{[O,M]}}([M,N]).$$

We will have proved the assertion of the theorem if we can find a point N such that

$$\mathcal{F}^{\Gamma_{[O,M]}}([M,N]) < \mathcal{F}^{\Gamma^i}([O,M]), \tag{4.3}$$

because then, for η small enough,

$$\frac{1}{\eta}\mathcal{F}_\infty^{\Gamma^i \cup [O,M]}(\eta[M,N]) < \mathcal{F}^{\Gamma^i}([O,M]),$$

where $\eta[M,N]$ is the η-homothetic of $[M,N]$ about M, hence, in view of the definition of $\mathcal{F}_\infty^{\Gamma^i \cup [O,M]}$ in Theorem 3.5,

$$\frac{1}{(1+\eta)}\mathcal{F}^{\Gamma^i}([O,M] \cup \eta[M,N]) < \mathcal{F}^{\Gamma^i}([O,M]).$$

But

$$\frac{1}{1+\eta}\mathcal{F}^{\Gamma^i}([O,M] \cup \eta[M,N]) = \mathcal{F}^{\Gamma^i}(\Sigma),$$

with

$$\Sigma := \frac{1}{1+\eta}\big\{[O, M] \cup \eta[M, N]\big\} \in \mathcal{A}^{\mathrm{O}}, \quad \text{and} \quad \mathcal{H}^1(\Sigma) = 1, \tag{4.4}$$

so that finally we obtain

$$-\mathcal{G}_1 \leq \mathcal{F}^{\Gamma^i}(\Sigma) < \mathcal{F}^{\Gamma^i}\big([O, M]\big) = -\mathcal{G}_{\text{clas}}.$$

The proof of (4.3) relies on "classical results" in the mathematical theory of stress intensity factors in fracture mechanics. It corresponds to the computation of the energy release rate associated to a straight line segment starting from the crack tip of a semi-infinite straight crack (Γ^i, or $\Gamma_{[O,M]}$). Set, as in Theorem 3.5,

$$\hat{u}^0 := \hat{w} + u^0_{\mathrm{O}},$$

as well as

$$\hat{\hat{u}}^0 = \hat{\hat{w}} + \hat{u}^0,$$

where $\hat{w}$, $\hat{\hat{w}}$ are respectively the functions that satisfies (3.17) for any $v \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus (\Gamma^i \cup [O, M]); \mathbb{R}^2)$, resp. $v \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus (\Gamma_{[O,M]} \cup [M, N]); \mathbb{R}^2)$, with $\epsilon(v) \in L^2(\mathbb{R}^2; \mathcal{S}^{2\times 2})$.

According to Dauge (1988), Grisvard (1989), the singular parts of $\hat{u}^0, \hat{\hat{u}}^0$ are respectively

$$u^0_{\mathrm{M}} = \sqrt{|x|}\big(K^{\mathrm{M}}_1 \phi^M_1 + K^{\mathrm{M}}_2 \phi^M_2\big), \qquad u^0_{\mathrm{N}} = \sqrt{|x|}\big(K^{\mathrm{N}}_1 \phi^N_1 + K^{\mathrm{N}}_2 \phi^N_2\big),$$

$|x|$ being this time the distance to the point M, resp. N, and ϕ^M_i, ϕ^N_i being functions of the polar angle with respect to the directions $\overrightarrow{OM}, \overrightarrow{MN}$, respectively. The stress intensity factors K^{M}_1, K^{M}_2, resp. K^{N}_1, K^{N}_2, are different from those in (2.2), but they satisfy

$$\big(K^{\mathrm{M}}_1, K^{\mathrm{M}}_2\big)^{\mathrm{T}} = F(\theta)(K_1, K_2)^{\mathrm{T}}, \qquad \big(K^{\mathrm{N}}_1, K^{\mathrm{N}}_2\big)^{\mathrm{T}} = F(\theta')\big(K^{\mathrm{M}}_1, K^{\mathrm{M}}_2\big)^{\mathrm{T}}, \tag{4.5}$$

where θ, θ' are the polar angles

$$\theta := \big(\widehat{\vec{e}_1, \overrightarrow{OM}}\big), \qquad \theta' := \big(\widehat{\overrightarrow{OM}, \overrightarrow{MN}}\big)$$

and the 2×2-matrix F is a universal analytic function of the polar angle (cf. Leblond 1989).

Remark 4.2 We wish to briefly elaborate on the universal character of the matrix F established in Leblond (1989). Clearly, the field u^0_{O} depends upon the elasticity of the material, as well as upon the boundary conditions imposed on $\partial\Omega$ from the outset. This is reflected in the values of the stress intensity factors K_1 and K_2. Then the solution to the problem for $\gamma^i \cup [O, M]$ is always given by (3.19) with $\hat{w}$ the solution to (3.17), and this independently of whether we started with Dirichlet boundary conditions, or any other kind of reasonable boundary condition on $\partial\Omega$. The solution $\hat{w}$ can be in turn decomposed as $\hat{w} = K_1\hat{w}_1 + K_2\hat{w}_2$ with obvious notation.

The stress fields associated with $\hat{w}_1$ or $\hat{w}_2$, are in turn independent of the elasticity of the material, or of the boundary condition imposed on $\partial\Omega$. Indeed, the stress fields σ_1, σ_2 associated to ϕ_1, ϕ_2 are so, thus the Airy functions ψ_1, ψ_2 associated with $\mathcal{C}\epsilon(\hat{w}_1)$, $\mathcal{C}\epsilon(\hat{w}_2)$ are biharmonic functions in $H^2_{\text{loc}}(\mathbb{R}^2 \setminus (\Gamma^i \cup [O, M]))$ with boundary conditions on $\Gamma^i \cup [O, M]$ that only depend on the angle θ. The precise boundary conditions are obtained upon replacing τ_{xx}, τ_{xy}, τ_{yy} by $\partial^2\psi/\partial y^2$, $-\partial^2\psi/\partial x\partial y$, $\partial^2\psi/\partial x^2$, respectively, in (3.26), with $C\epsilon(u^0_{\text{M}})$ replaced by, respectively, σ_1, σ_2, and by suitable integration by parts.

The determining fact that the Airy potential is biharmonic of course specific to 2d isotropic elasticity and the result would fail in any other context.

The computation of $\mathcal{F}^{\Gamma[O,M]}([M, N])$ is intimately connected to that of K^{N}_1, K^{N}_2. Indeed, according to Theorem 3.2 in Destuynder and Djaoua (1981), and because we now know, thanks to Dauge (1988), Grisvard (1989), that Assumption (H1) of that theorem is correct,

$$\mathcal{F}^{\Gamma[O,M]}\big([M, N]\big) = -C(\lambda, \mu)\big(\big(K^{\text{N}}_1\big)^2 + \big(K^{\text{N}}_2\big)^2\big), \tag{4.6}$$

where $C(\lambda, \mu)$ is an explicit positive function of the Lamé coefficients that depends on the adopted setting, i.e., plane strain, plane stress, or pure two-dimensional elasticity. (In Destuynder and Djaoua 1981, the adopted setting is plane stress.)

In view of (4.6), showing (4.3) amounts to showing that the maximum value of $(K^{\text{N}}_1)^2 + (K^{\text{N}}_2)^2$ is never obtained for $\theta' = 0$. Indeed, when $\theta' = 0$, Σ defined in (4.4) is precisely $[O, M]$, so that, if the maximal value is never attained at $\theta' = 0$, then that value is strictly greater than $-\mathcal{F}^{\Gamma^i}([O, M]) = \mathcal{G}_{\text{clas}}$, so that (4.3) must be satisfied.

The following expansion of $F(\zeta)$ as a function of ζ in a neighborhood of $\zeta = 0$ is derived in Amestoy and Leblond (1992):

$$\begin{cases} F_{11}(\zeta) = 1 - \frac{3}{8}\zeta^2 + \big(\frac{1}{\pi^2} - \frac{5}{128}\big)\zeta^4 + \big(\frac{1}{9\pi^4} - \frac{11}{72\pi^2} + \frac{119}{15360}\big)\zeta^6 + O(\zeta^8), \\ F_{12}(\zeta) = -\frac{3}{2}\zeta + \big(\frac{10}{3\pi^2} + \frac{1}{16}\big)\zeta^3 + \big(-\frac{2}{\pi^4} - \frac{133}{180\pi^2} + \frac{59}{1280}\big)\zeta^5 + O(\zeta^7), \\ F_{21}(\zeta) = \frac{1}{2}\zeta - \big(\frac{4}{3\pi^2} + \frac{1}{48}\big)\zeta^3 + \big(-\frac{2}{3\pi^4} + \frac{13}{30\pi^2} - \frac{59}{3840}\big)\zeta^5 + O(\zeta^7), \\ F_{22}(\zeta) = 1 - \big(\frac{4}{\pi^2} + \frac{3}{8}\big)\zeta^2 + \big(\frac{8}{3\pi^4} + \frac{29}{18\pi^2} - \frac{5}{128}\big)\zeta^4 + O(\zeta^6). \end{cases} \tag{4.7}$$

For $\theta' = 0$ to be a maximum of $((K^{\text{N}}_1)^2 + (K^{\text{N}}_2)^2)(\theta')$, we must have in particular that its derivative at 0 is 0. This implies that

$$\big(F'_{12}(0) + F'_{21}(0)\big)K^{\text{M}}_1 K^{\text{M}}_2 = 0,$$

or still, since $F'_{12}(0) + F'_{21}(0) = -1$,

$$K^{\text{M}}_1 K^{\text{M}}_2 = 0.$$

If $K_1^{\mathrm{M}}=0$, we must have $F_{12}^2(\zeta)+F_{22}^2(\zeta)\leq 1, \forall\zeta$, which is clearly not the case near 0 in view of (4.7), except if K_2^{M} is also 0. So, we may as well assume that $K_2^{\mathrm{M}}=0$.

But then, since M is also such that $\mathcal{F}^{\Gamma^i}([O,M]):=-\mathcal{G}_{\mathrm{clas}}$, we conclude that θ must satisfy

$$\begin{cases} F_{21}(\theta)K_1+F_{22}(\theta)K_2=0,\\ F_{11}'(\theta)K_1+F_{12}'(\theta)K_2=0,\end{cases} \tag{4.8}$$

the second relation holding true because the derivative of $((K_1^{\mathrm{M}})^2+(K_2^{\mathrm{M}})^2)(\zeta)$ must be 0 at $\zeta=\theta$. But, this is impossible, since $K_2\neq 0$, unless $\theta=0$, because by assumption, we assume the validity of the following conjecture.

Conjecture 4.3 *For all $\zeta\neq 0\in(-\pi,\pi)$, $F_{21}(\zeta)F_{12}'(\zeta)\neq F_{22}(\zeta)F_{11}'(\zeta)$.*

We have thus reached a contradiction upon assuming that the maximum value of $(K_1^{\mathrm{N}})^2+(K_2^{\mathrm{N}})^2$ is obtained for $\theta'=0$, unless $\theta=0$.

Now, if $\theta=0$, then 0 is a maximizer for $((K_1^{\mathrm{M}})^2+(K_2^{\mathrm{M}})^2)(\theta)$, so that, as before, $K_1K_2=0$, hence, since $K_2\neq 0$, $K_1=0$. But then, $(F_{12}^2+F_{22}^2)(\theta)=1+(3/2-8/\pi^2)\theta^2+O(\theta^4)$ must be maximal at 0, which is clearly not so. □

Remark 4.4 According to formulae (4.7), the following expansion holds true:

$$\frac{F_{21}(\zeta)F_{12}'(\zeta)}{F_{22}(\theta)F_{11}'(\zeta)}=1-\frac{8\zeta^4}{135\pi^2}+O(\zeta^6), \tag{4.9}$$

and thus Conjecture 4.3 is certainly verified when $\zeta\approx 0$. For large ζ's, the corroborating evidence is numerical: In Amestoy (1987), the author provides numerical values for the F_{ij}'s every 5° till 90°, then every 10° for larger angles, while Amestoy and Leblond (1992) give an expansion of those coefficients, up to order 21. The fit is rather impressive. For example, at $\zeta=\pi/2$, the values given in Amestoy (1987) are

$$F_{11}(\pi/2)=0.371,\qquad F_{12}(\pi/2)=-1.193,$$
$$F_{21}(\pi/2)=0.346,\qquad F_{22}(\pi/2)=-0.195,$$

while those given by the expansion in Amestoy and Leblond (1992) are

$$F_{11}(\pi/2)=0.3719,\qquad F_{12}(\pi/2)=-1.1935,$$
$$F_{21}(\pi/2)=0.3481,\qquad F_{22}(\pi/2)=-0.1966.$$

Using the expansion of Amestoy and Leblond (1992) in the interval $[-\pi/2,\pi/2]$, then curve-fitting with Amestoy (1987) for larger angles yields the numerical curves for the F_{ij}'s in Fig. 1.

Also, because of the expansions in Amestoy and Leblond (1992), F_{22} becomes 0 for $\zeta=\pm\zeta_2$ with $\zeta_2\approx 0.430\pi$, while F_{12}' becomes 0 for $\zeta=\pm\zeta_1$ with $\zeta_1\approx 0.425\pi$; note that $\zeta_1<\zeta_2$. The ratio $F_{12}'(\zeta)F_{21}(\zeta)/F_{11}'(\zeta)F_{22}(\zeta)$—an even function of ζ—is seen on Fig. 2 to be strictly decreasing as ζ grows from 0 to ζ_1. In the interval

Fig. 1 The F_{ij}'s computed

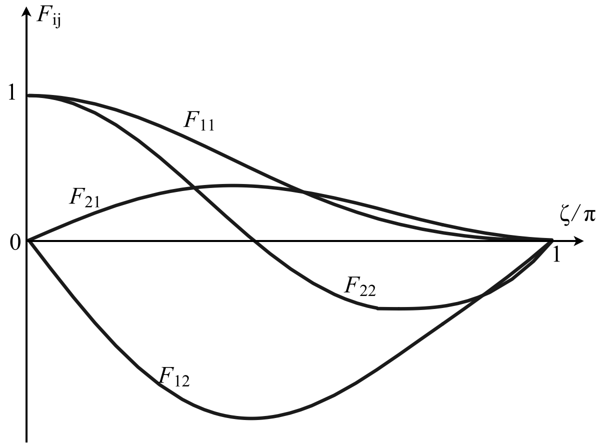
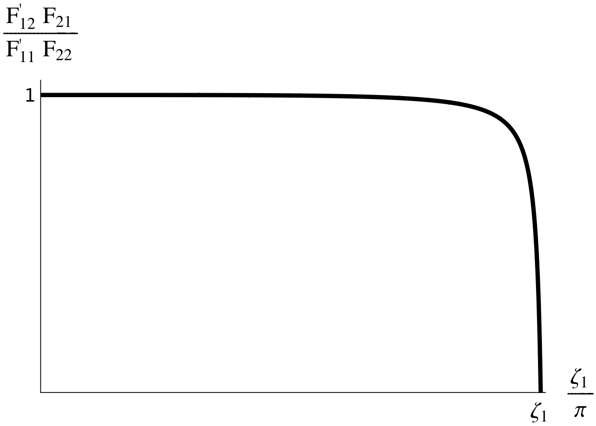


Fig. 2 Numerical check of the validity of Conjecture 4.3 on $[-\zeta_1, \zeta_1]$



(ζ_1, ζ_2), that ratio is negative and thus the conjecture must be satisfied there. Hence, it is satisfied on $(-\zeta_2, +\zeta_2)$.

The expansions in Amestoy and Leblond (1992) are not sufficient for an accurate computation of that ratio over the interval $[\zeta_2, \pi)$. However, a simple manipulation using the signs and parity properties of the F_{ij}'s, F'_{ij}'s would establish that there is no value θ outside the interval $(-\zeta_2, \zeta_2)$ that maximizes $(K_1^{\mathrm{M}})^2 + (K_2^{\mathrm{M}})^2$ and such that $K_2^{\mathrm{M}} = 0$.

Remark 4.5 Arguments very similar to those used in the previous proof would demonstrate that, when $K_2 = 0$, then, in the notation of that proof, $\theta = \theta' = 0$, provided that

$$\big(F_{11}(\zeta)\big)^2 + \big(F_{21}(\zeta)\big)^2 < 1, \quad \zeta \neq 0. \tag{4.10}$$

Once again, this is certainly true, according to (4.7), when $\zeta \approx 0$. For large ζ's, the corroborating evidence is, once again, numerical, cf. Fig. 3.

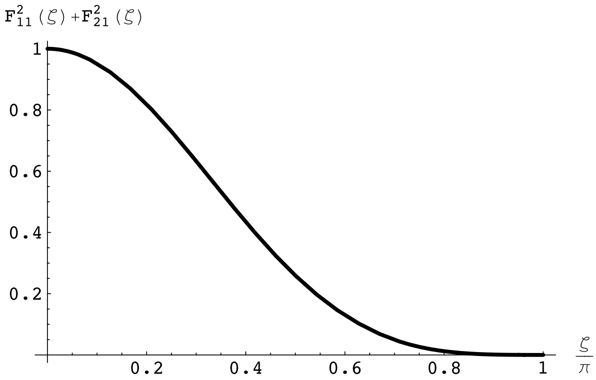


Fig. 3 Numerical check that $(K_1^{\mathrm{M}})^2 + (K_2^{\mathrm{M}})^2$ is maximal at $\theta = 0$ when $K_2 = 0$, see (4.10), according to the values in Amestoy (1987), Amestoy and Leblond (1992)

The answer to the second question formulated at the end of Sect. 3 is a bit more involved. Indeed, although we have just exhibited in Theorem 4.1 a set $\Gamma_\varepsilon = \varepsilon\Sigma$, with $\Sigma \in \mathcal{A}_1^{\mathrm{O}}$ which strictly decreases $\mathcal{F}^{\Gamma^i}$, so that, according to Theorem 3.1, for ε small enough, the energy increment rate

$$-\frac{1}{\varepsilon}\left\{\frac{1}{2}\int_\Omega \left(\mathcal{C}\epsilon\left(u^{\Gamma_\varepsilon}\right)\cdot\epsilon\left(u^{\Gamma_\varepsilon}\right) - \mathcal{C}\epsilon\left(u^0\right)\cdot\epsilon\left(u^0\right)\right)\mathrm{d}x\right\} > \mathcal{G}_{\mathrm{clas}}, \tag{4.11}$$

we have not produced a sequence $l_i \searrow 0$ with the ordering property (2.11) such that there exists an energy release rate for that sequence, and such that energy release rate strictly exceeds $\mathcal{G}_{\mathrm{clas}}$.

In order to reach a meaningful result, we must introduce a notion of metastability during a crack evolution. Specifically, in the setting of Sect. 2, assume that there exists a "smooth enough" evolution starting at γ^i. In other words, assume that the crack will extend from γ^i along a path Γ and that

S1. $\Gamma \in \mathcal{A}^{\mathrm{O}}$and satisfies Assumption A7;
S2. u^0 is a function of the time t, such that, if $\Gamma(t)$ denotes the add-crack at time t, and $l(t)$ its length, the following properties hold:
S3. $\Gamma(t) \in \mathcal{A}^{\mathrm{O}}$ and satisfies the ordering property in (2.11);
S4. $\Gamma(t) \subset \Gamma$;
S5. $l(0) = 0$ and $l(t)$ is continuous and strictly increasing in a neighborhood $[0, t_0)$, $t_0 > 0$, of 0.

Griffith's criterion for crack evolution states that, under such conditions, the energy release rate at time t, denoted by $\mathcal{G}(t)$, must be such that

$$\mathcal{G}(t) = k, \tag{4.12}$$

where k is a material characteristic sometimes called the fracture toughness. According to Lemma 2.6 and to Theorem 3.1, $\mathcal{G}(0)$ does exist and its value is less than or equal to $\mathcal{G}_{\mathrm{clas}}$.

Now, we adopt a "natural" metastability condition and refer the reader to Bourdin et al. (2008) for a discussion of the merits of such an assumption and of its relevance to classical fracture mechanics. See also Larsen (2010) for a different, but related approach to the stability of cracks.

Metastability *For all* $t \geq 0$, *there exists* $0 < \varepsilon_t \ll 1$ *such that*

$$\gamma^i \cup \Gamma(t) \quad \textit{minimizes} \quad \frac{1}{2}\int_{\Omega} \mathcal{C}\epsilon\big(u^{\Gamma}\big)\cdot\epsilon\big(u^{\Gamma}\big)\,\mathrm{d}x + k\mathcal{H}^1\big(u^{\Gamma}\big) \tag{4.13}$$

among all Γ*'s in* $\mathcal{A}^{\mathrm{O}}$, *with* $\Gamma \supset \Gamma(t)$ *and* $\mathcal{H}^1(\Gamma \setminus \Gamma(t)) \leq \varepsilon_t$. *In the last integral,* u^{Γ} *is a solution to the elastic equilibrium on* $\Omega \setminus (\gamma^i \cup \Gamma)$ *with boundary condition* $u \equiv u^0(t)$ *outside* $\overline{\Omega}$.

We use the above metastability at time $t = 0$. Since $\Gamma(0) = \emptyset$, then necessarily,

$$-\frac{1}{\mathcal{H}^1(\Gamma)}\mathcal{F}^{\gamma^i}(\Gamma) \leq k, \quad \forall \Gamma \in \mathcal{A}^{\mathrm{O}} \text{ with } \mathcal{H}^1(\Gamma) \leq \varepsilon. \tag{4.14}$$

However, choose $\Gamma = \Gamma_\varepsilon$ satisfying (4.11). Then with such an admissible Γ, we find that

$$\mathcal{G}_{\mathrm{clas}} < k,$$

so that, in particular

$$\mathcal{G}(0) < k,$$

a contradiction with Griffith's criterion. We have thus reached the following conclusion, which we state as a proposition.

Proposition 4.6 *If a crack evolution starting from the tip of* γ^i *satisfies metastability in the sense of* (4.13), *as well as Griffith's criterion* (4.12), *then it cannot satisfy Assumptions* S1 *to* S5.

In other words, assuming metastability, a connected add-crack cannot grow continuously in time along a path which has density $\frac{1}{2}$ at the crack tip. If it grows continuously in time it must grow along a crack with higher density (like a branching crack), or, if it grows along a crack of density $\frac{1}{2}$, it must do so brutally, i.e., with a jump in length at time $t = 0$.

It is not our purpose here to expound the consequences of this result and we refer the interested reader to Chambolle et al. (2009) for a detailed investigation of the impact of such a result. However, note that our result *prohibits, modulo metastability*, the coexistence of crack that would follow a "smooth" path *and* grow smoothly in time, which is precisely the starting point of most studies on crack kinking in fracture mechanics.

5 Concluding Remarks

As already discussed in the Introduction, this work is in part our contribution to the $G_{\max}$ versus $K_2 = 0$ debate which has been and remains a controversial topic in the fracture mechanics community. We do not reiterate the conclusions of the preceding section but wish to emphasize that those conclusions hinge on three ingredients.

First, we assume meta-stability of the crack at each time. Rejection of this postulate completely invalidates our contribution. Once again, meta-stability is a very widespread assumption in many fields of mechanics, starting with finite elasticity (Ball 1977), but commonality of belief is no voucher for the veracity of that belief.

Then we assume the truth of Conjecture 4.3. Numerical evidence, provided above, seems to corroborate the conjecture. Remark that, was this conjecture to be false, analyticity would imply that kinking can only occur along universal angles that would not depend upon the elasticity or the material, the shape of the domain, or the boundary conditions (see Remark 4.2). That would also fix the ratio K_1/K_2, provided that $K_2 \neq 0$ for those angles, and this independently of the same parameters.

Finally, our mathematical arguments are based on the definition of an energy release rate for add-cracks that live in a family of connected 1 dimensional sets of finite Hausdorff measure. This is the focus of Sect. 3 and represents the other part of the work presented in this paper. That contribution is independent of the first two ingredients and constitutes, to our knowledge, the first attempt at defining rigorously an energy release rate on a finite domain by connecting it, through blow up, to a computation on an infinite domain.

As a final note, the class of add-cracks for which our energy release rate is well defined is much larger that considered in the literature since it encompasses essentially all connected add-cracks of finite length. Of course, removing connectedness would be a great step forward. This would most likely entail a reformulation of the energy release rate solely in terms of the kinematic variable—the displacement u—that will then live in $SBD(\Omega)$, the crack being the jump set $S(u)$ of u, or more precisely its closure. None of the tools that have been developed in Chambolle (2003), Chambolle et al. (2008) and used in the present study apply; in particular, we do not even know that $\mathcal{H}^1(\overline{S(u)} \setminus S(u)) = 0$ in such a setting.

Acknowledgements The authors warmly thank Christopher Larsen for helpful and interesting discussions. The first two authors are partially supported by the Agence Nationale de la Recherche, MICA program ANR-08-BLAN-0082.

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